NAEP Mathematics Framework Update: Policy Discussion

Background

Each NAEP Assessment is guided by a framework that defines the knowledge and skills to be assessed at each grade. Through active participation of NAEP stakeholders, each framework is developed through a comprehensive process that considers various factors, such as state and local curricula and assessments, widely accepted professional standards, international standards, and exemplary research.

Framework development and update processes are overseen by the Assessment Development Committee (ADC). The ADC conducted a review of the 2017 NAEP Mathematics Framework (last updated in 2006), which included a discussion with external experts as well as a Board-commissioned inventory of state standards (sent to Board members under separate cover). Based on the ADC review, the Governing Board initiated an update of the framework. The Board awarded a contract to WestEd for implementation of the update project. WestEd has convened subject matter experts, practitioners, policy makers, administrators, researchers, business representatives, and members of the general public – serving as the Visioning and Development Panels in accordance with their Board-adopted Charge (Attachment D). The Charge calls for recommendations that balance necessary changes with the Board's desire for stable trend reporting, continued breadth of content coverage, and innovation.

Determining the content and format of each NAEP assessment is one of the Governing Board's Congressionally-mandated responsibilities. Using recommendations that reflect Visioning and Development Panel deliberations and public comment (Attachment A), the framework process concludes when the Governing Board adopts a framework that also reflects its concerns and priorities. At the August 2019 Board meeting, the full Board will discuss the April 2019 draft of the 2025 NAEP Mathematics Framework (Attachment C) and public comments on the draft (Attachment A) in preparation for Board action in November 2019.

Changing Landscape

A number of stakeholders, including but not limited to the Council of Chief State School Officers and the Council of the Great City Schools (Attachment B), have shared their concerns with Governing Board staff about the lack of alignment between state standards and the new draft NAEP Mathematics Framework – in particular regarding the inclusion of some data, probability, and statistics objectives in 4^{th} grade and a reduction in emphasis on algebra content in 8^{th} grade. States and TUDA districts want to ensure they receive the most relevant and useful information possible from NAEP; perceived misalignment can diminish NAEP's utility (and perhaps credibility) in those jurisdictions. Yet, NAEP's role is not merely to mirror state academic standards; it is to determine where the field currently is and anticipate where the field is going so that students have the opportunity to demonstrate mastery of these knowledge and skills at 4^{th}, 8^{th}, and 12^{th} grade. If the NAEP frameworks merely "follow" state standards, it could undermine the important audit function that has allowed NAEP to be a resource to states and the field.

The landscape in American education has changed significantly since the Board last updated the framework for NAEP mathematics. In 2006, state standards were all over the map – they varied widely in depth, organization, coherence, and rigor. The Board's job in that context was to make sense of the muddle: to determine what content was most important for students in elementary, middle, and high school to know and be able to do for success in their futures. Since then, however, states have significantly improved the quality of their standards, as documented by several organizations and experts. And there is strong consensus across more than 40 states that those higher quality standards should be taught in classrooms across the country.

This changing landscape alters the decision-making calculus for the Board. Consensus has largely endured over the last decade about the mathematics that students need to learn and in what general sequence; and those consensus standards are considered high-quality by respected independent experts. For the Board to make a decision to include content on NAEP assessments that few if any states include in their standards by that grade level will require extremely strong evidence, e.g., that leading math educators and mathematicians believe the current consensus standards are not serving children well; that the field of math is changing rapidly and will require new ways of teaching and learning; or that the highest performing nations are adopting new models of teaching and learning math that diverge from most states' current paths.

This positions decisions about the 2025 NAEP Mathematics Framework in a unique context. To maintain NAEP's role as a credible, independent source of information on student achievement, NAEP must be informed by leading edge states and countries. Given states' reform efforts, the challenge in today's context is to find the most salient ways for NAEP to continue its role of being a "North Star" of assessment for the country – while honoring and reflecting the work of leading states and maximizing the chance that trend can be maintained. This will allow NAEP to retain its role as the trusted independent measure of students' educational progress, reflecting what students have the opportunity to learn as well as what states want students to have the opportunity to learn.

2025 NAEP Mathematics Framework

At the August 2019 meeting, the Board will be asked to deliberate on how NAEP should confront the challenge of finding the most salient areas in which NAEP should be more forward-looking, while also determining areas where NAEP should honor and reflect the progress made in states. The discussion will assist the ADC in determining the Board's perspective in order to provide guidance to the Development Panel on key questions from the public comment:

1. ***Should NAEP only address what is covered in states?*** If the NAEP Mathematics Framework restricts to standards covered in states, what types of evidence should be used to include framework objectives that are only covered in a few states? Extensive public comment addressed areas of the framework that were intended to be forward-looking but are currently not covered in state standards. (See Chapter 2 summary below for details.)
2. ***How should NAEP use state assessments' emphases to inform how NAEP emphasizes content areas?*** The NAEP Mathematics Framework includes 5 content areas: Number Properties and Operations; Geometry; Measurement; Data Analysis, Statistics, and Probability; and Algebra. Each area has different weights depending on the grade level. What types of rationales should be used as a basis to change how the content areas are weighted, especially considering that changes could impact trend reporting?
3. ***Are there any remaining concerns that the Framework Panel should address to allow for Board adoption of the framework?*** Beyond the previous issues, are there other issues that the Board would like to see addressed in the final framework draft in November 2019?

Public Comment on the Draft 2025 NAEP Mathematics Framework

A total of 1,856 statements were submitted as public comments by 109 individuals, which included 10 states and several organizations. Over 600 individuals in 49 of the 50 states downloaded the draft from the project website. Overall, over 20 organizations were engaged in discussions on the draft framework.

Across the entire framework, 10% of comments were praise for the draft. About 38% of comments requested edits to clarify presentation (7%) or meaning (24%) or to make additions (5%) or deletions (2%). Approximately 26% of comments were suggestions on how to prioritize content or practices. The remaining input, about 20%, was scattered across topics, some beyond the scope of the framework.

Chapter-Level Summary of Public Comments

Below is an annotated version of the table of contents for the draft framework, presenting public comments by chapter.

The majority of public comment on Chapter 1 requested editing to clarify ideas (about 40%). The next most common type of feedback was praise for the ideas in the chapter, particularly attention to opportunity to learn and changes in state standards (about 25%). Remaining comments focused on technology students use in and out of school (including calculators) and the potential pros and cons of digital assessment and scenario-based tasks.

Most comments about the *Background on NAEP* section were requests for clarification (40%). A few comments asked about the purpose of NAEP (e.g., to what extent it might be related to what gets taught) and some suggestions were made for additional resources to be consulted.

Comments on the *Visioning and Development Process* section were requests for clarification (50%) or praise for tackling opportunity to learn (30%).

Many comments on the section on *Major Changes* were praise (33%), largely for the attention to practices and contextual variables. Most of the remaining comments were requests for clarification about terminology or main ideas that are be addressed in detail later in the framework (40%) [Note: The Framework Development Panel is currently drafting crisp statements about mathematics literacy and NAEP mathematical practices to clarify these main ideas of the framework].

The *Changes from the 2009-2017 Framework* section presented rationales for the major updates of the framework. Most comments on this section called for greater clarity and justification in the offered rationales. The Framework Development Panel will revise the table to improve clarity and reflect updated chapter revisions. For instance, the forward-looking aspects of the draft framework were largely based on the rapidly growing need for data literacy, as well as for spatial reasoning and related skills. There was also a review of the mathematics needed to successfully engage in the Next Generation Science Standards.

The highest proportion of comments on this chapter (about 40%) were calls for revisiting the prioritization of objectives across the grade levels. The instructions for submitting public comment specifically requested that reviewers generate comments about the priority of math topics. The prioritization comments (*n*=364) generally did not question the value of content objectives but suggested that objectives apply at different grades than they appeared in the draft. About a quarter of the prioritization comments focused on the tension between opportunity to learn given current state standards and specific objectives, particularly two grade 4 objectives in Data Analysis, Statistics, and Probability – both addressing Probability topics. Far fewer comments were offered about any other objectives (e.g., three comments about the use of "proportionality" in a Grade 4 Algebra objective). About 15% of prioritization comments suggested revision of the Algebra objectives in grade 8 to more closely reflect the language of college and career ready standards. For grade 12, three stakeholders noted that the increasing prevalence of STEM and non-STEM pathways in high school could encourage additional objective-level revisions.

About 10% of comments on Chapter 2 concerned the balance of content areas at each grade (i.e., what percent of items is devoted to each content area). The current NAEP Mathematics Framework and the draft 2025 framework both specify five content areas: Number Properties and Operations; Geometry; Measurement; Data Analysis, Statistics, and Probability; and Algebra. The draft 2025 framework decreases the balance of the assessment devoted to Algebra at grade 8 to 25% (from 30%), and increases the balance for Data Analysis, Statistics, and Probability to 20% (from 15%). Public feedback was split on whether or not it was appropriate to make this balance adjustment in grade 8.

Comments seeking clarification made up almost 30% of response to Chapter 2. These comments, particularly about the rationales for changes, included requests for more information on the nature and use of mathematical literacy in the objectives and on the assessment.

An additional 10% of Chapter 2 comments were suggested additions or deletions; 5% mostly about *adding* Data Analysis, Statistics, and Probability objectives and another 5% were about things to delete (again, public feedback was divided, many of the suggestions for deletions were about the same objectives or topics for which additions were suggested). As with the prioritization calls for revisiting grade level placements of objectives, the panel is looking closely at the suggestions regarding additions and deletions as it reconsiders the content objectives in each grade in light of the purpose of NAEP (to reflect current student achievement and anticipate near-future changes in educational focus).

An additional 5% of comments offered praise for edits seen as responsive to changes in state standards over the last decade and to the importance of mathematical literacy.

The highest proportion of comments on Chapter 3 (about 50%) were calls for clarification of ideas or terminology. Many of these comments related to whether or not the term "practice" was helpful, given that the term is used by states to refer to college and career ready standards. Most of these comments sought distinctions among the NAEP mathematical practices and more examples of each practice at each grade.

Unlike the current NAEP Mathematics Framework's "mathematical complexity" construct, the draft 2025 framework attends to the cognitive process dimension of the assessment through five mathematical practices: Representing; Abstracting and Generalizing; Justifying and Proving; Mathematical Modeling; and Collaborative Mathematics. Comments reflected overwhelming support for the introduction of the practices – no stakeholder rejected them. While there were some calls for clarifications in how the practices are articulated, there was also overwhelming support for the five mathematical practices for the 2025 NAEP Mathematics Assessment and beyond – one organizational stakeholder argued for adding "attending to precision" (one of the eight Standards for Mathematics Practice in college and career ready standards), while one stakeholder argued for deletion of Collaborative Mathematics as a practice. Several stakeholders asked for a crosswalk between the five practices in the draft 2025 framework and practices articulated in other standards. They also asked for more explanation on why these five practices were appropriate for NAEP, and there was a request for clarification that these practices are not intended as instructional practices.

The draft 2025 framework removes the construct of mathematical complexity that exists in the current NAEP Mathematics Framework. While several stakeholders supported efforts to refine the complexity construct, they also wanted the framework to provide more information about how variations in cognitive demand would be reflected in the cognitive process dimension of the assessment (i.e., how knowledge and skills will become more sophisticated as content objectives and practices are simultaneously used to construct assessment items).

As with the Chapter 2 comments, public response was divided (for each of the 5 practices, some comments suggested very low or no priority while others asserted high priority for the same practice). Almost 20% of Chapter 3 comments were assertions about the relative merits of the various practices and related assessment balance. There was extensive support for Collaborative Mathematics as a practice – 16 stakeholders explicitly affirmed the need to assess this practice on NAEP (more than any other practice), though a few other stakeholders raised potential accessibility concerns. As noted above, one stakeholder requested its removal. A few comments argued for grade-level distinctions of the balance of the assessment devoted to each practice.

About 10% of comments were suggested additions or deletions (8% about additions, many suggesting "problem solving" as a practice; 2% about deleting one or more of the practices). The panel is currently discussing plans for communicating about the NAEP mathematical practices as being in service of problem solving, not distinct from it.

A little over 10% of comments offered praise for edits seen as responsive to the changes in state standards over the past 40 years.

Many comments on chapter 4 (about 40%) sought clarification about tools, item types, and item formats; many asked for more insight about scenario-based tasks.

Another 22% of Chapter 4 comments suggested much more information was needed on why the proportions for the use of item types, particularly the balance of scenario-based tasks was warranted.

About 15% of Chapter 4 comments applauded the framework's attention to access and equity. Comments also cautioned about assuring the appropriate use of technology on the assessment and ensuring that accessibility goals of the framework are met.

Comments seeking more detail (particularly detail for item development that has traditionally been in the *Assessment and Item Specifications*) made up about 12% of response to Chapter 4.

Most comments on Chapter 5 (about 40%) made assertions or suggestions about reporting. Some comments offered observations about how various contextual variable questions might be perceived by families, while others asked for specifics on how the answers to the questions would inform NAEP reporting. Another subset of comments was about how opportunities to learn might be used to report results and how NAEP might differentiate its reporting for different stakeholders.

Another 20% of comments were about the priorities within reporting (e.g., high priority contextual information vs. low priority – the instructions to commenters included a specific request for such review).

About 20% of comments were praise on the diversity of perspectives in the Framework Development Panel and the clarity and attention to detail of the narrative in the chapter (e.g., in distinguishing purposes and roles of NAEP).

Just under 20% of comments were requests for clarification, most having to do with how to interpret the Achievement Level Descriptions (ALDs).

Few comments were offered on the appendices (14 total). Most were suggestions for clarifying the language of the ALDs and their use in reporting.

June 7, 2019

National Assessment Governing Board
U.S. Department of Education
800 North Capitol Street NW – Suite 825
Washington, DC 20002-4233
Attention: Michelle Blair, Assistant Director (Assessment Development)

Dear Ms. Blair:

As the organizations representing state and district education leaders across the country, the Council of Chief State School Officers (CCSSO) and the Council of the Great City Schools (CGCS) work together closely on many issues that impact state and local education systems. Yet we do not always agree and often arrive at our decisions based on different perspectives. Today, we have come together to provide our feedback on the April 18, 2019 draft of the *Mathematics Framework for the 2025 National Assessment of Educational Progress*.

Both of our organizations recognize the role NAEP plays to help state and district leaders understand student academic progress both within their respective states and districts and across the nation, with NAEP serving as the nation's report card. We appreciate the thoughtfulness with which the National Assessment Governing Board (NAGB) approaches all decisions regarding NAEP and your willingness to engage with both CCSSO and CGCS directly and with our members on this draft framework. At the same time, from our different perspectives at the state and local levels, we both share similar concerns with the current draft framework.

Thank you for hearing us on this critical issue. Please don't hesitate to reach out with any questions.

Sincerely,

Carissa Moffat Miller
Executive Director
Council of Chief State School Officers

Michael Casserly
Executive Director
Council of the Great City Schools

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Attachment B

June 7, 2019

National Assessment Governing Board
U.S. Department of Education
800 North Capitol Street NW – Suite 825
Washington, DC 20002-4233
Attention: Michelle Blair, Assistant Director (Assessment Development)

Council of the Great City Schools Comments on the Draft Mathematics Framework for 2025 National Assessment of Educational Progress

Introductory Comments

The Council of the Great City Schools (Council), a coalition of the nation's largest central city school districts, submits the following comments on the Draft Mathematics Framework for the 2025 National Assessment of Educational Progress (NAEP). Over the years, the Council has worked closely with the National Assessment Governing Board (Governing Board) and the National Center for Education Statistics (NCES) on a variety of efforts to measure and improve student learning outcomes. Therefore, the Council supports the Governing Board's efforts to periodically review and update NAEP policies and practices.

The Council is dedicated to the improvement of education for children in the nation's inner cities. The Council and its member districts work to help our public-school children meet the highest standards and become successful and productive members of society. The organization and the 27 Trial Urban District Assessment (TUDA) participants regularly use results from NAEP to measure our progress in achieving our goals. In fact, the Council of the Great City Schools initiated TUDA in 2000 as a way of holding ourselves and our students to the highest standards. As a result, we are heavily invested in any changes in policies and practices related to NAEP and other national measures of educational progress.

Given state autonomy in setting expectations for standards, curriculum, and assessments, NAEP serves the vital role of providing an assessment framework for cross-state comparisons. The Council's comments in this submission are focused on ensuring that NAEP remains a rigorous assessment; that it addresses the national shift college- and career-readiness standards have brought to the knowledge and skills students are exposed to in elementary, middle, and high school; and that it accurately measures the academic trends of students across the country based on what they have had an opportunity to learn.

The comments that follow are organized by chapter in the proposed NAEP framework. For each chapter we provide our perceived strengths of the document and specific areas that we feel need improvement. If a comment(s) could be stated for multiple chapters, we listed both in parentheses, i.e., opportunity to learn (chapter 1, 5). In some instances, an area of strength or area needing improvement was connected to a specific statement in the draft framework, so we included the actual text in italics or bold. Where necessary, we also provide background support for our suggested changes or revisions.

These comments are followed by a summary of the major issues we have identified, and recommendations for the Governing Board and Visioning Panel.

Broadly, we believe that the new framework fails to reflect the general shift in state standards most states have adopted. This problem is particularly acute in grade four mathematics. Specifically, we recognize the importance of students mastering statistics, data analysis, and probability content prior to completing their secondary education. However, given recent changes in the sequencing of learning objectives, the new framework for grade four mathematics assesses content that many students across the nation will not learn until grade six or beyond. Consequently, the nation, states, and districts may begin to question whether the NAEP performance trends reflect tangible changes in student achievement or content differences between what is taught and the NAEP item pool.

Chapter 1. Overview

Areas of Strength

- Chapter One of the draft framework provides a clear and detailed overview of the purpose and importance of the National Assessment of Educational Progress

- This chapter incorporates the latest information from educational research, policy, and practice into the development of the NAEP Mathematics Assessment Framework

- The chapter expands the view of "achievement gaps" to emphasize "opportunity to learn gaps," a directive of the Visioning Panel. The Opportunity to Learn language incorporates relevant research and articulates a clear expectation that all students have access to the NAEP Mathematics Assessment content (chapters 1, 5).

    - *A major goal in the process was to ensure that NAEP is designed and implemented in ways that allow all students to show their best work in terms of what they know and can do mathematically. This means ensuring maximum accessibility to different groups of students who live and learn in a wide range of contexts. (page 1, draft framework)*

    - *The Visioning Panel's directive to develop an expansive conception of opportunity to learn (see Exhibit 1.2)… This view, too, is now gaining more traction in research, practice, and policy. These bodies of research informed the Development Panel's conception of opportunities to learn. In particular, when results are interpreted in ways that emphasize achievement gaps without attending to opportunity gaps, differences in subgroups of students can be misinterpreted as differences in student ability, rather than differences due to unequal and inadequate educational opportunities. (chapter 1, 5)*

- The draft framework maintains the use of the well-understood achievement level descriptors Basic, Proficient, and Advanced, which are used in previous frameworks. This includes providing examples of what these levels of achievement look like for specific grade bands as well as specific topics in the achievement level descriptions (examples provided in Chapter 5)

- The draft framework includes mathematical literacy for grade 12 in the draft 2025 NAEP Framework. This addresses the need for students to use and apply mathematics in a variety of careers, as well as for informed citizenship, personal financial management, consumer related decision-making, and for discerning bias. (*Mathematical literacy is integrated and highlighted as a cross-cutting theme).*

- The new framework eliminates the confusing rating methodology found in previous versions about mathematical and cognitive complexity, instead emphasizing mathematics content, cognitive demand, and mathematical practices.

- The new draft framework is more consistent with college- and career-readiness standards in emphasizing student-centered learning over teacher-centered instruction. The new draft frameworks are also more connected to the work of NGSS and NCTM in their emphasis on mathematical practice.

- The new framework eliminates two objectives in the area of data analysis, statistics, and probability: (a) list all possible outcomes of a given situation or event, and (b) represent the probability of a given outcome using a picture or other graphic. The framework also adds an objective that is consistent with state standards: recognize and generate simple equivalent (equal) fractions and visually explain why they are equivalent. *From Exhibit 1.4. Comparison of 2009, 2017 and 2025 NAEP Mathematics Frameworks, including the following statement: Objectives for grade 4 were undated to reflect changes in what students have an opportunity to learn by grade 4.*

- Finally, the framework advocates for different item formats and assessment design, including introducing new scenario-based tasks. This will allow students to demonstrate their understanding of content while applying one or more mathematical practices. *As indicated earlier in the document, "this will allow mathematical processes and the interacting social and mental activities of knowing and doing mathematics to become visible."*

Areas of Concern

- From Exhibit 1.1. Guidelines from the Visioning Panel: Mathematics and Test Design/Technology (page 4) includes language that *the mathematics content of the preK-12 curriculum has significantly evolved, and these changes need to be reflected in NAEP. We recommend a broadening of the content in several ways, including (b): a re-examination of statistics, data analysis and probability concepts and skills in light of current scholarship and standards.*

  This bullet indicates that the framework development teams planned a re-examination of statistics, data analysis, and probability concepts and skills in light of current scholarship and standards. However, it is not clear that this has happened, or that this review was sufficiently thorough, since a re-examination would have revealed that at grade 4 these content areas are not prevalent in most state assessment documents.

- From Exhibit 1.1 Guidelines from the Visioning Panel: Mathematics and Test Design/Technology (page 4) includes *attention to a wider range of technological tools available for students. Moreover, under test design and technology, the STRATEGIC USE OF TECHNOLOGY, includes a recommendation that NAEP revisions leverage technology to increase the assessment's authenticity (allowing students to use the technologies they use in and out of school).* This is a worthy goal, but it is not clear that the frameworks don't bias the test in favor of students who have regular access to the technology that NAEP will be using.

Chapter 2. Mathematics Content

For this chapter, our comments are primarily focused on **areas needing improvement** and specifically address grade four. While we comment on several strands or areas, our focus for grade four is in the areas of Data Analysis, Statistics, and Probability. Please note that we are not advocating for the removal of Statistics, Data Analysis, and Probability from the draft framework. Instead, we are questioning the grade level at which these skills are assessed.

Areas of Strength

- Some standards for grade 4 have been updated to reflect changes in what students have an opportunity to learn by grade 4, e.g., based on *current research, national and international assessment frameworks, national standards, and state standards more broadly.*

- The draft 2025 NAEP framework includes mathematical literacy for grade 12. This addresses the need for students to use and apply mathematics in a variety of careers, as well as for informed citizenship, personal financial management, consumer related decision-making, and for discerning bias. (*Mathematical literacy is integrated and highlighted as a cross-cutting theme in chapter 1, 2).*

- Most of the objectives in the 2025 draft framework at grade 8 are rigorous and appropriate for the grade level.

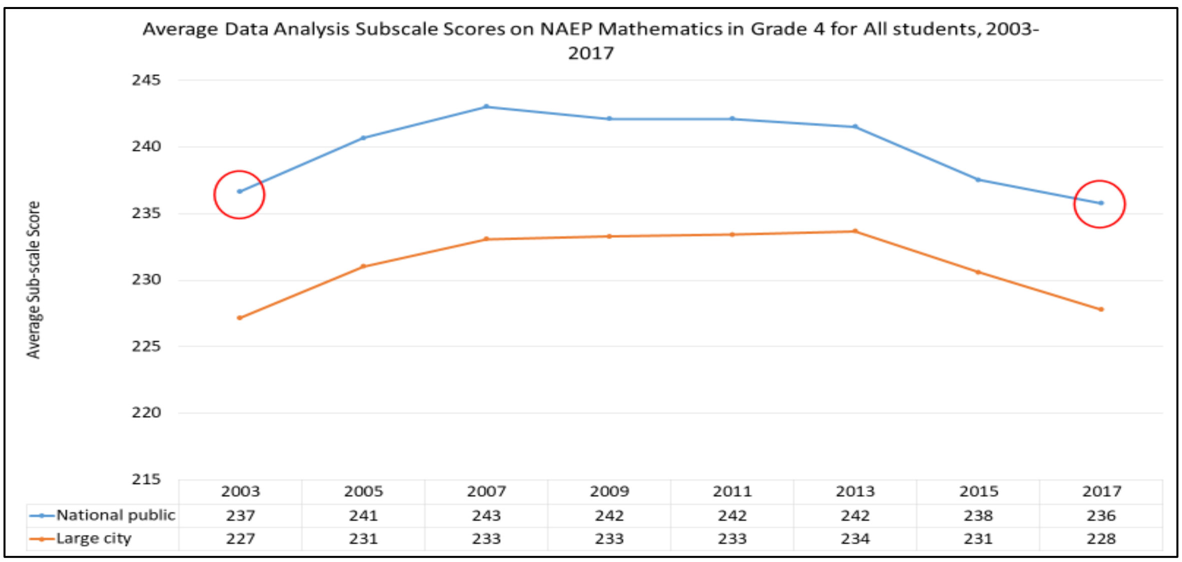
Areas of Concern

- Even though the teams proposed revisiting standards for grade four (From Exhibit 1.4. Comparison of 2009–2017 and 2025 NAEP Mathematics Frameworks), it appears that the Objectives for grade 4 were not fully updated to reflect changes in what students have the opportunity to learn by grade 4.

  The narrative in the draft includes an in-depth and thoughtful discussion about moving away from a narrow and technical view of achievement gaps to a broader lens focused on opportunity to learn. In particular, the framework states, "*when results are interpreted in ways that emphasize achievement gaps without attending to opportunity gaps, differences in subgroups of students can be misinterpreted as differences in student ability, rather than* *differences due to unequal and inadequate educational opportunities.*" This includes the mathematics content that students have access to or the opportunity to learn by grade 4.

This is more than an idle academic observation. This mismatch in what students are assessed on and what they have had the opportunity to learn is evident in the current frameworks, which were written before the standards used now by most states were crafted. Scores may have been affected in ways that the new draft frameworks now propose to perpetuate. For example, the Council examined results on the Data Analysis and Probability strand and found that the mismatch in what is assessed and what students have had an opportunity to learn is nationwide in scope and is found in every student group examined, including students scoring at advanced levels. As Figure 1 illustrates, after climbing six scale score points between 2003 and 2007, scores for the nation dropped six scale score points on data analysis over the last two assessment cycles in 2015 and 2017 after college- and career-readiness standards had become more fully implemented across the country.

Figure 1. Average Data Analysis Subscale Scores on NAEP Mathematics in Grade 4 for All Students, 2003-2017



The new draft frameworks for 2025 in grade 4 include not only data analysis but also statistics and probability, geometry, and other areas before the state standards presume they are taught. Examples include--

Data analysis, statistics, probability

- Given a set of data or a graph, describe the distribution of data using median, range, mode, or shape. This objective initially appears in grade 6 in state standards.
- In state standards, by grade 4 students are expected to analyze data and connect it to operations/algebraic thinking. For example, students are expected to solve one- and two-step "how many more" or "how many less" problems using information presented in bar

graphs, pictographs, etc.

- o Use informal probabilistic thinking to describe chance events (i.e., less likely and more likely, certain and impossible). This standard initially appears in grade 7 in state standards. For example, consider deleting items such as Exhibit 3.11, Grade 4 NAEP Probability Spinners Item.

- o Determine a simple probability from a context that includes a picture. This standard initially appears in grade 7 in state standards.

Geometry

- o Identify the images resulting from flips (reflections), slides (translations) or turns (rotations). This standard initially appears in grade 8 of state standards.

Number properties and operations (ratios and proportional reasoning)

- o Recognize or describe a relationship in which quantities change proportionally. This standard initially appears in grade 7 of state standards. The development committee might consider changing this standard to: "Recognize or describe the difference between a relationship that is either additive or multiplicative." This would be appropriate for students in grade 4 as they consider the distinction between an additive relationship or a multiplicative relationship."

- o Use simple ratios to describe problem situations. This initially appears in grade 6 in some state standards.

- The *Compare Two Sets of Related Data* objective under Data Analysis, Statistics, and Probability in grade 4 needs greater specificity so that it doesn't exceed what students have had the opportunity to learn by grade 4. For example, the objective does not specify whether students will compare data using measures of central tendency or range or whether they should be making graphical comparisons.

- Prior to grade 8 in state standards, identifying and representing functional relationships is confined to linear relationships. There is limited work with nonlinear functions, including quadratic functions by grade 8. The focus in grades 6 through 8 is on exploring linear functions, solving linear equations, and inequalities, while quadratics is the major focus in the algebra category for high school.

- The percentage of items devoted to algebra under the new draft framework has declined by five percentage points. The Council believes that it is important to nurture students who are algebraically capable as they pursue STEM fields and have an assessment that measures their progress at grade 8.

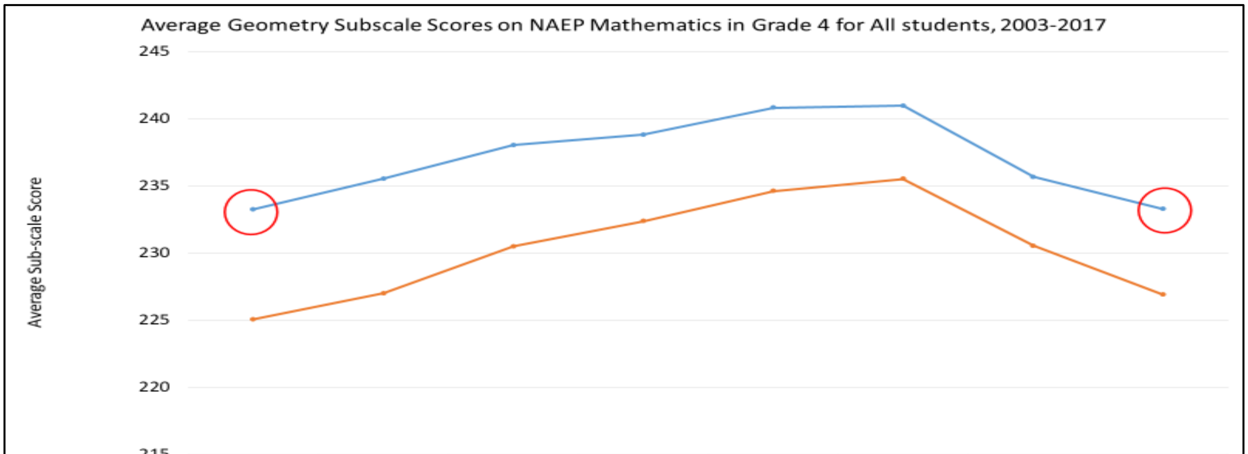
- The proportion of college-and career-readiness state assessments that cover Geometry and Data Analysis at grade 4 is small or non-existent according to the work of Daro, et al. Yet, the framework committee did not make the adjustments cited in the proportion of Geometry and Data Analysis, Statistics, and Probability concepts and skills proposed for the 2025 NAEP Framework—NAEP coverage of Geometry and Data Analysis is currently about 30 percent of the assessment. (See Figure 2).

- *At grade 4, some content in the 2017 NAEP objectives was not regularly part of schooling until grade 6 (Daro et al., 2015; Hughes et al., 2013; Johnston et al., 2018). To address this, six objectives were removed at grade 4 where grade 8 objectives were similar and more appropriately timed to assess students on mathematics they would have had a chance to learn. Also, attention in early grades to equation as an equivalence between two values led to the addition of one objective in grade 4 Algebra. Research suggested that no other objective was absent from NAEP that was commonly assessed in states (Johnston et al., 2018), from pages 16-17 of the 2025 Draft NAEP Mathematics Framework.*

Figure 2. [REDACTED; CONVEYS EMBARGOED RESULTS]

- It is important to address an apparent contrast between Geometry and Data Analysis. As discussed previously, Daro et al. found clear evidence that key aspects of Data Analysis were not addressed until after grade 4 in the common core standards. The mismatch between the NAEP framework and state standards is therefore clear-cut. However, a cursory review of state standards reveals that Geometry is, in fact, a part of the standards at and prior to grade four. However, upon closer inspection, Geometry was only an "additional focus area" in state standards—meaning it is not an area that is emphasized in the standards and was only marginally assessed on state college- and career-readiness tests. This means that, like Data Analysis, Geometry is not a topic area that is consistently or comprehensively taught at or prior to grade four. It is therefore no surprise that the Council's analysis of Geometry subscale performance across time shows that the nation declined eight scale score points since 2013 – even more than the six-scale score point decline in data analysis shown earlier. (Figure 3)

Figure 3. Average Geometry Subscale Scores on NAEP Mathematics in Grade 4 for All Students, 2003-2017



	2003	2005	2007	2009	2011	2013	2015	2017
National public	233	236	238	239	241	241	236	233
Large city	225	227	230	232	235	236	231	227

Chapter 3. Mathematical Practices

Areas of Strength

- This chapter on Mathematical Practices reflects the most current research focusing on how students learn and develop a deeper understanding of mathematical concepts, skills, and applications. This includes recognizing that mathematics is a social construct that requires constructive engagement in mathematics and builds on students' social and cultural knowledge and life experiences. This allows students to develop not only a conceptual understanding of the mathematics that they are learning but also the belief that mathematics is worthwhile, sensible, and feasible.

  - *Over the last two decades mathematics education research has experienced a "social turn" (Lerman, 2000), marked by a shift toward investigating the learning of mathematics in relation to social activity (Adler, 1999; Bell & Pape, 2012; Black, 2004; Civil & Planas, 2004; Ernest, 1998; Enyedy, 2003; Moschkovich, 2007, 2008; NCTM, 1991; van Oers, 2001). A mathematical practice represents what the community writ large values in the patterns of activity that one engages in when doing mathematics. Practices are not at the margins of mathematics. They are – along with content – at the core of mathematics (chapter 3, page 45*).

- This chapter also presents assessment items that connect mathematical content to the mathematical practices. This includes specific examples in Exhibit 3.20. Practices and Content Table, (pgs. 71-74).

- Some mathematical practices are connected directly in the 2025 NAEP Mathematics Framework to the appropriate practice in Principles and Standards 2000 (NCTM) or the eight mathematical practices in state standards.

- The draft framework also features the addition of collaborative mathematics as a practice, reflecting the growing work on growth mindset in mathematics. Moreover, this connects to the standard *Make sense of problems and persevere in solving them* found in state standards.

Areas of Concern

- Not all NAEP mathematical practices have been linked or tied to state standards or to the Principles and Standards 2000 (NCTM). It was helpful to the reader when this connection was made in the draft frameworks for the practice of mathematical modeling. For example, consider connecting or relating Mathematical Practice 2 (abstracting and generalizing) to "Look for and making use of structure" and "Look for and express regularity in repeated reasoning" in state standards.

Chapter 4. Overview of the Assessment Design

Areas of Strength

- This chapter of the draft framework includes different item formats and assessment design like scenario-based tasks. This will allow students to demonstrate their understanding of content while applying one or more mathematical practices.

  - *Scenario-based tasks have both context and extended storylines to provide opportunities to demonstrate facility with mathematical practices.*

Chapter 5. Reporting Results of the NAEP Mathematics Assessment

Areas of Strength

- Opportunity to Learn incorporates relevant research and provides explicit language and expectations that all students have access to the NAEP Mathematics Assessment (chapters 1, 5).

  - *As noted in Chapter 1, research has informed an expanded view of the factors that shape opportunities to learn, including time, content, instructional strategies (e.g., how students are grouped for learning; the mathematical tasks they engage in; the opportunities students have to reason, model, and debate ideas), and instructional resources (e.g., the qualifications of their teachers; the material resources available to them; classroom and school policies for the mathematics that students have access to), pgs. 91-92.*

  - *Students who see themselves and who are seen by others as capable mathematical thinkers are more likely to participate in ways that further their learning; students who do not see themselves and are not seen by others as capable mathematical thinkers are likely to be disengaged.*

- The framework connects *Opportunity to Learn* with existing contextual variable items that include, time, content, instructional strategies, and instructional resources.

- The framework proposes the development of additional items in the areas of student engagement and identity, views of mathematics teaching and learning, features of classroom instruction, and engagement in mathematics in and out of school.

Areas of Concern

- The development committee is proposing to add family engagement variables to the background sections of NAEP. Consider the impact of including Family Engagement in Mathematics (OTL-IR) as one of the contextual variables. On its face, there may not be a problem with this, but we can foresee how the results could be misused or misinterpreted by people who might claim based on the data that some parents don't care about their children's education.

Discussion of Major Issues

1. The inclusion of some data analysis, probability, statistics items in the draft framework is at odds with the aspiration of the National Assessment Governing Board and its Visioning Panel to incorporate more focus on opportunity to learn. In fact, assessing material before students have been exposed to the specified concepts--by definition--undercuts the very concept of opportunity to learn.

2. Members of the framework development team have argued that material related to data analysis, probability, and statistics should be taught and assessed at or before grade 4. This is not advisable for several reasons. One, it is almost always poor practice and policy to have assessments and test content drive or define content or curriculum. Decades of experience suggest that assessment should follow content, not define it. Two, the aspirations of the development committee appear to be at odds with the statute authorizing NAEP, which states that "Any assessment authorized under this section shall not be used by an agent or agents of the Federal Government to establish, require, or influence the standards, assessments, curriculum, including lesson plans, textbooks, or classroom materials, or instructional practices of States or local educational agencies." Three, the inclusion of concepts on NAEP that have not been taught has not been vetted through national, state, and local debate with the thoroughness that was conducted during the introduction of the college- and career-readiness standards and its derivatives. Four, the public recognizes that NAEP is the "gold standard" in assessment, which the Council would not dispute. This reputation is based largely on the assessment's psychometric excellence. At the same time, being the gold standard does not suggest an immutable assessment of content that does not change with developments in the field. For example, the recent transition to a digitally based assessment illustrated the board's desire to ensure that the test remain the gold standard. We would argue that updating the assessment to reflect new instructional standards is another essential change that must be made for NAEP to remain valid and relevant. The development panel risks tarnishing the test's sterling reputation by measuring material that is not taught.

3. The inclusion of material before it has been covered in the nation's classrooms is likely to increase inequity in the nation's educational system because the students who will do well on this material are students privileged enough to encounter the concepts outside of school. This is likely to widen achievement gaps rather than narrow them.

4. The current mismatch between the NAEP framework and state standards will inevitably lead to confusion and misinformation. In fact, the public and the press is likely to conclude that math performance is declining when, in fact, lowered scores may be the result of test misalignment. A recent example might illustrate the point about how this works currently. On June 5, 2019, a principal researcher at the American Institutes of Research wrote an article published in *The 74* titled "Did Common Core Standards Work? New Study Finds Small but Disturbing Negative Impacts on Students' Achievement." If an NCES contractor can mistake an alignment problem with a performance problem, it will not take the public and the press long to jump to the wrong conclusion as well. The result will not only be a public that cannot tell whether their schools are getting any better, but it could easily result in misinformed policies to correct issues that may not exist.

5. The nation's major cities volunteered to take NAEP in order to determine if we—the nation's largest urban school systems—were making progress with all the reforms we were pursuing. We also wanted the ability to compare ourselves to similar districts across state lines, something that no other assessment allowed. And we wanted a way to figure out what was working in our districts and what wasn't. Over the years, we have used the results to achieve these goals. However, if NAEP ceases to provide useful, valid data on district progress, there will no longer be a reason for districts to continue to participate in the Trial Urban District Assessment of NAEP.

Recommendations

The Council recommends that the Governing Board and Development Panel—

- ❖ Determine what areas are not covered by grade 4 in most state standards and move them to grade 8.

- ❖ Determine the weight or relative importance of specific objectives on most state assessments and realign NAEP accordingly at tested grade levels.

- ❖ Ascertain the joint effects of grade level misalignment of objectives and the relative weights of those objectives to determine the impact on past and future student scores. Adjust objectives and previous scores accordingly to aid the interpretation of future scores and improve the accuracy of the trend line.

- ❖ Take the time necessary to fully review the objectives and make alterations in the framework, even if it means delaying the release of the framework.

- ❖ Maintain the trend lines in the state NAEP results to ensure continuity of results.

CCSSO
Council of Chief State School Officers

June 7, 2019

National Assessment Governing Board
U.S. Department of Education
800 North Capitol Street NW – Suite 825
Washington, DC 20002-4233
Attention: Michelle Blair, Assistant Director (Assessment Development)

Dear Ms. Blair:

The National Assessment of Educational Progress (NAEP) has long played an important role in education in the United States. As the organization representing state K-12 education leaders across the country, the Council of Chief State School Officers (CCSSO) recognizes the role NAEP plays to help state and district leaders understand student academic progress both within their respective states and districts and across the nation, with NAEP serving as the nation's report card. We appreciate the thoughtfulness with which the National Assessment Governing Board (NAGB) approaches all decisions regarding NAEP and your willingness to engage with CCSSO directly and with our members. We would also like to thank you for the opportunity to provide feedback on the April 18, 2019 draft of the *Mathematics Framework for the 2025 National Assessment of Educational Progress*.

We know that one of the primary reasons to update the NAEP Mathematics Assessment is to ensure it is informed by recent standards, curricula, and instruction. Since the Mathematics Framework was last updated, state leaders have made fundamental changes to their state content standards and how those standards are taught, which includes both what is taught and the sequencing of what is taught. While each state makes its own determination about its standards and has adopted standards that fit its unique needs, as a nation we have seen more consistency in state standards than we had in the past as all states have transitioned to college- and career-ready standards based in research. These standards have taken hold in states and remain consistent. The opportunity to update the NAEP Mathematics Framework is well timed; we have much more information about state standards and have had an opportunity to analyze how what is taught in classrooms aligns to what is assessed on NAEP. The Achieve report, *Is NAEP Out of Step with State Standards?*, summarizes some of the key studies and identifies specific areas where there is a disconnect between the current NAEP assessment and what is happening in schools.

While we appreciate the work that has gone into updating the Mathematics Framework, we are concerned about two outstanding issues the current draft does not address. The first is related to content that will be tested, based on the new Framework, at Grade 4 but no longer taught in or before Grade 4 in the majority of states and districts. For example, the Framework suggests that probability will be tested on Grade 4 NAEP. In most states, probability is now introduced in Grade 7. As a result, students would be tested on mathematical concepts they have not the opportunity to learn, which renders NAEP much less meaningful. We appreciate that the role of NAEP is an independent measure of student achievement. As part of its responsibility in administering NAEP, the National Center for Education Statistics is required to ensure that NAEP "cannot be used to establish, require, or influence the standards, assessments, curriculum, or instructional practices of states or local educational agencies" (as set forth on the NAEP website). We believe our comments on this issue reinforce that responsibility.

Our second major concern is based on the proportion of items proposed in Grade 8. Specifically, the draft Mathematics Framework indicates that Data Analysis, Statistics, and Probability was increased by 5% (to 20%), and Algebra was decreased by 5% (to 25%). We are concerned that the decreased emphasis on Algebra is not reflective of the value many states are placing on Algebra. Again, with this disconnect, the relevance of NAEP for state leaders would decrease. We strongly encourage you to revisit the alignment issue broadly, based on these examples, for the draft Mathematics Framework and consider additional changes.

We recognize that information about the long-term trend is extremely important to both NAGB and to state and district education leaders. We do not think the additional changes we are recommending need to disrupt that trend. As we have seen over the years as changes have been made to NAEP, including the most recent move to a fully online assessment, NAGB is able to both address the concerns we note above and maintain the long-term trend.

Thank you for engaging with state education leaders on this critical issue.

Sincerely,

Carissa Miller

Carissa Moffat Miller
Executive Director
Council of Chief State School Officers

Mathematics Framework for the 2025 National Assessment of Educational Progress

Draft – 4/18/2019

Developed by WestEd
Under contract to the National Assessment Governing Board
Contract # 91995918C0001
Project Website: <https://www.naepframeworkupdate.org>

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Chapter 1

OVERVIEW

The National Assessment of Educational Progress (NAEP) has measured student achievement nationally since 1973, and state-by-state since the early 1990s, providing the nation with a snapshot of what students in this country know and can do in mathematics. Urban school districts that meet certain selection criteria can volunteer to participate in the Trial Urban District NAEP Assessment.

The major purpose for this framework is to identify what should be measured at grades 4, 8, and 12. For each subject area measured by NAEP, a framework is used to describe the subject matter to be assessed, the assessment questions to be asked, as well as the assessment's design and administration. Frameworks are designed to inform NAEP assessment development. The most recent updates of the NAEP Mathematics Framework were completed in 2001 for grades 4 and 8, and in 2006 for grade 12. These updates were reflected in the 2005 and 2009 NAEP Mathematics Assessments, respectively.

This framework offers guidance for how developments in educational research, policy, and practice over the past two decades should be reflected in the NAEP Mathematics Assessment. This updated framework is based on a visioning and development process that engaged curriculum experts, researchers, assessment experts, teachers, and other leading educators. A major goal in the process was to ensure that NAEP is designed and implemented in ways that allow *all* students to show their best work in terms of what they know and can do mathematically. This means ensuring maximum accessibility to different groups of students who live and learn in a wide range of contexts—urban, rural, or suburban; who bring a wide spectrum of experiences, backgrounds, and needs; and who represent a wide range of communities of different ethnic, cultural, and linguistic strengths and "lived experiences," both inside and outside school. This emphasis is based on a growing awareness in research and practice of the significance of these differences in teaching and learning.

There are several important audiences for this framework. Primary among these are the educators in schools, policymakers, students and their families, and the general public. In addition, this framework and the accompanying NAEP Mathematics Assessment and Item Specifications is directed toward the National Center for Education Statistics (NCES) and their contractors, critical NAEP partners, who will use this framework to develop the 2025 NAEP Mathematics Assessment.

Background on NAEP

There are two distinct components to the NAEP Mathematics Assessment which differ in purpose. The NAEP Long-Term Trend (LTT) assessment measures long-term trends in achievement among 9-, 13-, and 17-year-old students nationally. This unique measure allows for comparisons of students' knowledge of mathematics since NAEP was first administered in 1973 and its framework has been unchanged ever since. The second assessment, referred to as "main NAEP," is adjusted over time to reflect shifts in research, policy, and practice. The main NAEP assessment is administered at the national, state, and selected urban district levels. In mathematics, NAEP results are reported on student achievement in grades 4, 8, and 12 at the national level, and for grades 4 and 8 at the state level and for large urban districts that volunteer to participate. The content and format of the main NAEP Mathematics Assessment is the focus of this framework.

Taken together, the NAEP assessments provide a rich, broad, and deep picture of patterns in U.S. student mathematics achievement. National and state level results are reported in terms of scale scores, achievement levels, and percentiles. These reports provide comprehensive information about what U.S. students know and can do in mathematics. In addition, NAEP provides comparative subgroup data according to gender, race/ethnicity, socio-economic status, geographic region; describes trends in performance over time; and reports on relationships between student achievement and certain contextual variables. All of this information is essential for understanding what students have had an opportunity to learn.

The main NAEP assessment is administered to a nationally representative sample of students to report on student achievement in the aggregate. The assessment is not designed to measure the performance of any individual student or school. To obtain reliable estimates across the population that is assessed, a large pool of assessment items is developed. Subsets of items from the large pool are selected to administer to each student in the sample. Student results on the main NAEP assessments are reported for three achievement levels established and defined by the National Assessment Governing Board (Governing Board), which oversees NAEP:

- *NAEP Basic* denotes partial mastery of prerequisite knowledge and skills that are fundamental for performance at the *NAEP Proficient* level.
- *NAEP Proficient* represents solid academic performance for each NAEP assessment. Students reaching this level have demonstrated competency over challenging subject matter, including subject-matter knowledge, application of such knowledge to real-world situations, and analytical skills appropriate to the subject matter.
- *NAEP Advanced* signifies superior performance beyond *NAEP Proficient*.

Examples of what these levels of achievement look like for specific grade bands and specific topics are provided in the achievement level descriptions later in this document. Chapter 5 includes further discussion of these achievement levels.

This document describes an assessment framework, and not a curriculum framework. It lays out the basic design of the assessment by describing the mathematics content and mathematical practices that should be assessed and the types of questions that should be included. It also describes how various assessment design factors should be balanced across the assessment. In broad terms, this framework attempts to answer the question: What mathematics knowledge, skills, and practices should be assessed on NAEP at grades 4, 8, and 12? This document does not attempt to answer the questions: What mathematics should be taught? Or by what pedagogical methods? Moreover, the framework does not cover all relevant content but instead was developed with the understanding that some concepts, practices, and activities in school mathematics are not suitable to be assessed on NAEP, although they may well be important components of a school curriculum.

The Visioning and Development Process

The process for updating the mathematics assessment framework consisted of the review by experts in mathematics education research, policy, and practice representing key stakeholder groups. This process – which is described in the Governing Board (2018) *Framework Development Policy Statement* – involved visioning for the update, and then development.

The Visioning Panel is tasked with formulating "high-level guidance about the state of the field to inform the process, providing these in the form of guidelines." The specific charge included:

> The Visioning and Development Panels will recommend to the Governing Board how best to balance necessary changes in the NAEP Mathematics Framework at grades 4, 8, and 12, with the Governing Board's desire for stable reporting of student achievement trends and assessment of a broad range of knowledge and skills, so as to maximize the value of NAEP to the nation; and the Panels are also tasked with considering opportunities to extend the depth of measurement and reporting given the affordances of digital based assessment.

The 30-person Visioning Panel met in November 2018 to determine principles, goals, and policies to guide the Mathematics Framework update. During this meeting, the Visioning Panel learned about NAEP, the framework update process, and available NCES resources. Using this information, the panelists identified and discussed issues related to developments in mathematics education research, policy, and practice that should inform the design of the assessment framework. The Visioning Panel then developed a set of guidelines for the recommended updates to the assessment framework. The Guidelines were clustered in three domains – mathematics, assessment design and technology, and opportunities to learn – and are summarized in Exhibits 1.1 and 1.3.

Exhibit 1.1. Guidelines from the Visioning Panel: Mathematics and Test Design/Technology

MATHEMATICS
1. EXPANSION OF ATTENTION TO STUDENT REASONING AND MATHEMATICAL PRACTICES **We recommend** defining mathematical practice constructs of priority interest in the framework (e.g., representing, abstracting and generalizing, justifying and proving, modeling, mathematical collaboration), providing examples of how they can be assessed (e.g., in the Assessment and Item Specifications), and using these definitions to systematically assess these practices, integrated with content, in 2025.
2. SIGNIFICANT BROADENING OF MATHEMATICAL DOMAINS AND COMPETENCIES The mathematics content of the preK-12 curriculum has significantly evolved, and these changes need to be reflected in NAEP. **We recommend** a broadening of the content in several ways, including:

(a) content that reflects research on culturally relevant, responsive, and sustaining pedagogies, ethnomathematics, neurodiversity, and students' funds of knowledge;
(b) a re-examination of statistics, data analysis and probability concepts and skills in light of current scholarship and standards documents;
(c) attention to a wider range of technological tools available for students;
(d) highlighting foundational mathematical themes that cut across different areas of content domains (e.g. geometry, algebra) and the grade bands from 4^{th} to 8^{th} to 12^{th} grades; and
(e) consideration of a new cross-cutting theme or content area (at grade 12) that expands on calculus-readiness and statistics to include increasingly relevant applied mathematics important to informed citizenship, to personal financial and other decisions, and a variety of careers.

3. ATTENTION TO THE BALANCE OF COGNITIVE DEMAND
Currently, different levels of "mathematical complexity" in NAEP afford a balance between low-level items that ask for recall or demonstration of procedures, medium-level items that require connection-making on multi-step procedures, and high-level items that require analysis, creativity, synthesis, or justification and proof. **We recommend** a NAEP mathematics framework update in terms of cognitive demand.

TEST DESIGN AND TECHNOLOGY

4. TEST DESIGN
We recommend the integration of content and practice skills through leveraging interactive multimedia scenario-based tasks as a way to provide more authentic tasks for students to complete (e.g., NAEP Technology and Engineering Literacy (TEL) tasks).

5. STRATEGIC USE OF TECHNOLOGY
We recommend that NAEP revisions leverage technology to increase the assessment's authenticity (allowing students to use the technologies they use in and out of school) and the assessment's accessibility. Given the digital divide, as the NAEP instrument evolves, panels should address known and potential implementation issues and recommend ways to mitigate issues of access and test-taking that could occur in under-resourced communities.

The third domain of guidelines concerned students' opportunities to learn (see Exhibit 1.3). The Visioning Panel's conception of opportunity to learn was informed by educational research on students and their in- and out-of-school learning and experiences, as well as research on the variations in resources that shape what students have an opportunity to learn about mathematics in the United States (e.g., Cohen, Raudenbush, & Ball, 2003).

Opportunity to learn is generally understood to refer to inputs and processes that shape student achievement, including the school conditions, curriculum, instruction, and human, material, and social resources to which students have access. When opportunity to learn was first used as a concept, Carroll (1963) emphasized "the amount of time allowed for learning" (Carroll, 1989, p. 26). For the past 50 years, the concept of opportunity to learn has continued to evolve, as have efforts to measure in-school opportunities to learn, with the majority of scholars focusing on the classroom as the unit of analysis and instruction as central. Indicators have been clustered in various ways (e.g., Abedi & Herman, 2010; Elliott & Bartlett, 2016; Herman, Klein, & Abedi, 2000; Husén, 1967; Schmidt, Burroughs, Zoido, & Houang, 2015; Wang, 1998). These can be grouped into four strands: time, content, instructional strategies, and instructional resources. Examples of indicators that have been used in research are provided in Exhibit 1.2.

Exhibit 1.2. Opportunity to Learn (OTL) Strands

OTL Strand	*Example indicators*
Time	time scheduled for instruction proportion of allocated time used for instruction time students are engaged in learning time students are experiencing a high success rate of learning
Content	content exposure content emphasis content coverage
Instructional Strategies	instructional practices (e.g., providing feedback, direct instruction, group work, extended projects, problem-based learning) classroom climate instructional group size cognitive expectations for student learning
Instructional Resources	Teacher quality (e.g., teacher educational background, teacher knowledge, teaching experience, beliefs about students, beliefs about mathematics) Material resources (e.g., textbooks, manipulatives) School policies (e.g., tracking) *Students' experiences, out-of-school learning, and funds of knowledge*

Missing from the traditional framing of OTL is acknowledgment that students themselves are a resource in learning, including their interests, abilities, and in- and out-of-school experiences. Yet research suggests that students' experiences out-of-school can be directly relevant to their abilities to think mathematically and use mathematics (e.g., Martin, 2000; Nasir & Hand, 2008). Some scholars refer to this as students' "funds of knowledge," defined as the skills, knowledge, habits of mind, practices, and experiences acquired through historical and cultural interactions of an individual in their community, family life, and culture through everyday living (e.g., Civil, 2016; de Freitas & Sinclair, 2016; González, Moll, & Amanti, 2005; Moll, Amanti, Neff, & González, 1992). While this knowledge might differ from those of the teacher or the traditional curriculum, educators can tap into the broad experiences of students to make powerful connections that enable learning and thus can be understood as an additional resource in instruction and assessment. As a result, the Development Panelists added the indicator "*Students' experiences, out-of-school learning, and funds of knowledge"* to the OTL instructional resource category.

The Visioning Panel's third domain in the guidelines concerned opportunities to learn and students' opportunities to demonstrate what they have learned (see Exhibit 1.3).

Exhibit 1.3. Guidelines from the Visioning Panel: Opportunities to Learn and to Demonstrate Learning

OPPORTUNITIES TO LEARN AND
OPPORTUNITIES TO DEMONSTRATE LEARNING

6. EXPANSIVE CONCEPTION OF OPPORTUNITIES TO LEARN
We recommend developing a broad approach to framework update that scaffolds attention to opportunities to learn mathematics content, processes, and practices. This intent should be woven into the objectives in the framework, the item types and examples, and realized in contextual variables used on surveys.

Contextual Variables. **We recommend** updates to contextual variables in surveys should include attention to students' views of mathematics, and of themselves as mathematics learners; students' views of their peers', teachers', and school's beliefs/interest in their progress in mathematics; students' views of mathematics teaching and mathematics assessment (including NAEP); student access to and engagement with the language and culture of the test; teachers' knowledge of what has been taught before NAEP is administered; and teachers' beliefs about mathematics, mathematics teaching, and what their students can do.

7. ACCESSIBLE ASSESSMENTS FOR ALL STUDENTS
We recommend developing authentic assessment items with multiple access points that provide diverse populations of students with opportunities to demonstrate their mathematical knowing and reasoning in creative, authentic ways. This includes improving the accessibility of the assessment through short term goals like reconsidering test time limits, establish testing conditions that are more closely aligned with learning conditions (the use of typical tools, for example, or allowing teachers to be present) as well as longer term efforts to document how the current assessment remains inaccessible. Items should have consequential validity, be engaging to students, reflect guidelines for "low floor, high ceiling" tasks that provide opportunities for multiple approaches, and connect to students' lived experiences and funds of knowledge. Making the testing technologies widely available to students and teachers well before the assessment would also increase access and authenticity.

The full set of guidelines was passed on to the Development Panel, a subset of 15 Visioning Panelists who were tasked with developing drafts of updated project documents and engaging in the detailed deliberations about how issues outlined in the Visioning Panel discussion should be reflected in a recommended framework. The three documents include: a recommended framework, assessment and item specifications, and recommendations for contextual variables that relate to the subject being assessed. The Development Panel convened three 2-day meetings to prepare these three documents, as well as two webinars to prepare for and review progress. In between and after meetings, the Development Panel drafted and revised documents. The updates included responding to the guidelines enumerated above. In addition, all updates were made in congruence with Governing Board policies. The Development Panel drew on a wide range of policy and research documents to inform its deliberations, including expert review of the NAEP Mathematics Framework commissioned by the National Assessment Governing Board (2018).

Complementary to the Visioning and Development Panels, a Technical Advisory Committee (TAC) of eight recognized measurement experts advised the panels with regard to technical issues. The TAC met six times and representatives attended the panel meetings. The TAC consulted with Development Panelists to specify recommendations for content and cognitive dimensions in the framework, as well as recommendations for item and assessment design.

Major Changes in this Framework

Given the Visioning Panel guidelines for the framework update and the research on mathematics teaching and learning, technology and assessment, and curriculum and assessment standards, this NAEP Mathematics Framework reflects several major changes (see Exhibit 1.4 at the end of this chapter for a comprehensive review of changes). Major changes are summarized below.

Mathematics Content

Chapter 2 presents an updated distribution of content domains (e.g., Data Analysis, Statistics, and Probability; Geometry; Algebra) at grades 4, 8, and 12. As recommended by the Visioning Panel guidelines, for grade 12 aspects of mathematical literacy are integrated and highlighted as a cross-cutting theme to include increasingly relevant applied mathematics important to informed citizenship, personal financial and other decisions, and a variety of careers.

Mathematical Practices

Earlier frameworks differentiated between three levels of “mathematical complexity”: low, moderate, and high. These levels are difficult to operationalize because complexity can be manipulated by varying non-mathematical dimensions, such as working memory or attentional demands, rather than complexity within the domain of mathematics itself. Current efforts to define mathematical complexity have emphasized instead attention to “higher-order thinking” or “mathematical reasoning,” as well as a delineation of mathematical activity as “practices” or “processes.” Given these developments, this framework eliminates the language of mathematical and cognitive complexity and includes instead attention to both mathematics content and mathematical practices, building on this most recent work.

Since the late 1980s, there have been ongoing efforts to more clearly specify mathematical processes like “higher-order thinking” or “mathematical reasoning.” Current conceptions of mathematical knowledge and skill have shifted to mathematical practices or processes. For example, in *Adding It Up*, the National Research Council (2001) enumerated five strands of mathematical mastery, including

- *conceptual understanding*—comprehension of mathematical concepts, operations, and relations
- *procedural fluency*—skill in carrying out procedures flexibly, accurately, efficiently, and appropriately
- *strategic competence*—ability to formulate, represent, and solve mathematical problems
- *adaptive reasoning*—capacity for logical thought, reflection, explanation, and justification
- *productive disposition*—habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one’s own efficacy.

The National Council of Teachers of Mathematics (NCTM) discusses five "mathematical processes standards": problem solving, reasoning and proof, communication, connections, and representation (NCTM, 2000). The language of "practice" has become increasingly popular, establishing a foothold through the publication of both the Common Core State Standards for Mathematics (CCSS-M; National Governors Association Center for Best Practices, Council of Chief State School Officers, 2010) and the Next Generation Science Standards (NGSS, 2013), as well as in discussions of core teaching practices (NCTM, 2014). The popularity of the term does not mean that there is universal agreement on its meaning. This framework defines mathematical practice as routines, norms, processes, or habits of mind that are needed to do the work of mathematics. In other words, the mathematical practices arguably shift the customary, habitual, or expected teacher-centered way of doing mathematics to student-centered sense-making approaches. Based on the current state of the field, this framework identifies five mathematical practices for the NAEP Mathematics Assessment:

Mathematical Practice 1: Representing
Mathematical Practice 2: Abstracting and Generalizing
Mathematical Practice 3: Justifying and Proving
Mathematical Practice 4: Mathematical Modeling
Mathematical Practice 5: Collaborative Mathematics

These mathematical practices are described in depth in Chapter 3 (see also Exhibit 3.20 for a crosswalk of mathematics content and mathematical practices).

Mathematical Literacy

The framework addresses the issue of mathematical literacy. The definition used for this framework is rooted in national and international educational and assessment policy (e.g., OECD, 2018, p.17). For the NAEP Mathematics Assessment, mathematical literacy is the application of numerical, spatial, or symbolic mathematical information to situations in a person's life as a consumer, worker, or citizen.

Although mathematical literacy is a capacity that applies to students of all ages, it is of particular concern with regard to the grade 12 mathematics for NAEP. For instance, given the fact that high school students do not all pursue advanced study in mathematics, the NAEP Mathematics Assessment should assess grade 12 students' mathematical literacy. However, there exists no comprehensive conceptualization of mathematical literacy that delineates the ways in which students might use mathematics in informal settings to make decisions as citizens, voters, consumers, or employees.

To address this issue, mathematical literacy is conceptualized in this framework as a cross-cutting theme that has the potential to be relevant to all of the mathematics content and mathematical practice domains described. To signify particularly appropriate content areas and practices, see the symbol # in the exhibits found in Chapters 2 and 3.

Item Formats and Technology in Assessment

A fourth major change involved item formats and the role of technology in assessment. As noted above and as further explained in Chapter 4, technological innovation is relevant to NAEP both because it allows for more authentic assessments and because it allows for a broader range of accommodations to meet students' needs.

Since 1992, the NAEP Mathematics Assessment has used three formats or item types: multiple choice, short constructed response, and extended constructed response. In 2017, the NAEP assessment began to include these item types in a digital platform, as part of the NAEP transition to digital-based assessment. The transition to digital administration provided opportunities to expand the range of formats used for items.

In advancing the expansion of item types and formats, several themes emerged. One theme concerns how research on the ways the knowledge that students possess from in- and out-of school experiences can be used to design assessments that capture their capacity to do mathematics. This includes the use of interactive, multimedia ("next generation") scenario-based tasks to assess what students know and can do. By expanding item types and thoughtfully using technology, the aim is to provide greater access to all students, as well as to diversify the ways in which students' achievement can be recognized and measured.

A second theme concerns the use of technology to permit assessment of mathematical practices, including an expanded range of response types leveraging tool-based and discourse responses. Less often noted but equally important was a third theme concerning the intended or unintended negative consequences of technology, which include inequitable daily access to technologies. That is, while technology may have the potential to increase access and opportunities to demonstrate learning, students unfamiliar with technologies used in the assessment could be at a disadvantage.

Expansive Understanding of Contextual Variables

A final major shift in this framework involves an expanded conceptualization of opportunities to learn and relevant contextual variables. What students learn is inseparable from the conditions of their mathematical learning and broader social aspects of mathematics learning. In particular, this framework articulates an expansive conception of opportunities to learn, including the addition of students and their experiences as an instructional resource. We propose the addition of some math-specific questionnaire survey items.

Overview of Framework Chapters

Chapter 2 describes the content domains, including number properties and operations (including computation and understanding of number concepts); measurement (including use of instruments and concepts of area and volume); geometry (including spatial reasoning and applying geometric properties); data analysis, statistics, and probability (including graphical displays and statistics); and algebra (including representations and relationships). Each content area is broken into subtopics (e.g., for number properties these are number sense, estimation, number operations, ratios and proportional reasonings, and properties of number and operations) in an exhibit identifying what students should know and be able to do at grades 4, 8, and 12.

Chapter 3 describes the mathematical practices – the working practices of *doing* mathematics – that students must also master in order to fully understand mathematics and to demonstrate mathematics achievement. These include representing, abstracting and generalizing, justifying and proving, mathematical modeling, and collaborative mathematics. The chapter argues that content and practices are interwoven and interdependent: one cannot demonstrate mathematics achievement without knowing content and being able to think mathematically. Chapter 3 also offers example items across Grades 4, 8, and 12 that illustrate how mathematical practices would be contextualized in particular content.

Chapter 4 focuses on issues of technology and accessibility, as well as assessment design and item format, which are intricately connected to technology and tool design. The chapter argues for the need to ground the NAEP Mathematics Assessment in tasks in authentic and familiar contexts to foster student engagement with the assessment. Second, by expanding item types and thoughtfully using technology, the NAEP Mathematics Assessment can provide greater access to all students, diversify the ways in which student achievement can be recognized and measured, and more robustly assess both what students know and can do. This will involve expanding on a wider range of item types including scenario-based tasks, along with a subset of existing discrete NAEP items that capture student understanding of content and mathematical practices.

Chapter 5 addresses how NAEP results are reported. The chapter describes the three NAEP achievement levels, as well as the development of the mathematics achievement level descriptions (see Appendix A). The chapter lays out an expansive conception of "opportunity to learn" as called for by the Visioning Panel Guidelines and informed by recent research. The chapter discusses how research on student diversity and schooling informs mathematics-specific contextual variables.

Changes from the 2009-2017 Framework

Exhibit 1.4 compares this framework for the 2025 NAEP Mathematics Assessment and those used for the 2009–2017 NAEP Mathematics Assessment.

Exhibit 1.4. Comparison of 2009–2017 and 2025 NAEP Mathematics Frameworks

Topic	Change	Rationale
Mathematical Content	● The grade 4 version of six objectives were removed (two objectives each in Number and Operations; Geometry; and Data Analysis, Statistics, and Probability). One objective was added to grade 4 in Algebra.	● Objectives for grade 4 were updated to reflect changes in what students have an opportunity to learn by grade 4. Research across NAEP, state standards, and national policy documents (Daro, Hughes, & Stancavage, 2015; Hughes, Daro, Holtzman, & Middleton, 2013; Johnston, Stephens, & Ratway, 2018) indicated that some content in the grade 4 areas is not regularly part of

Mathematical Content (continued)		schooling until grade 6. Associated objectives were removed at grade 4 because grade 8 objectives were similar and more appropriately timed to assess students on mathematics they have had a chance to learn. One addition at grade 4 was in response to developments in early grades (across states) related to equation as an equivalence between two values. Research comparing state standards to objectives for NAEP did not suggest other cross-state consensus on additional objectives nor removal of objectives beyond what is already in NAEP (Johnston et al., 2018).
	● Three grade 8 objectives were edited, one was deleted in Number and Operations, and one was added in Algebra.	● Objectives for grade 8 were updated to reflect shifts in expectations evident from research reviews of state and national standards.
	● Descriptions of objectives in grade 12 edited. In Measurement, one objective made optional and one new optional objective added.	● Objectives for grade 12 were updated based on recent research on expectations for mathematical literacy.
	● The five mathematical practices in Chapter 3 address the mathematical activity in the subtopic of "mathematical reasoning" (this subtopic was introduced in 2009 for Number and Operations; Geometry, Data Analysis, Statistics, and Probability; Algebra). The objectives in the mathematical reasoning subtopic have been removed.	● With the introduction of mathematical practices (see Chapter 3), mathematical reasoning was no longer needed as a subtopic. To preserve attention to the content that was uniquely present in some of the mathematical reasoning objectives, objectives in other subtopics were edited (e.g., Number and Operations subtopic 3e in grades 4 and 8 was "Interpret…" and is now "Interpret, explain, or justify… ." For full details on all changes, see the *Assessment and Item Specifications*).

Mathematical Content (continued)	● Distribution of items for each content area at grades 4 and 12 remains the same. In grade 8, the proportion of items in Data Analysis, Statistics, and Probability was increased 5% (to 20%) and for Algebra decreased by 5% (to 25%).	● Adjustment to distribution of items in grade 8 reflects the increase in attention to data-driven mathematics skills in late elementary and middle grades. Research across state standards did not indicate any adjustment in distributions was needed for grades 4 or 12 (Johnston et al., 2018).
Mathematical Complexity	This was a chapter on the cognitive complexity involved in the doing of mathematics. It defined mathematical complexity as "the demands on thinking that an item expects" (NAGB, 2017). The chapter was replaced by a new chapter, Mathematical Practices.	Many decades of research and development in the processes of knowing and doing mathematics have indicated that assessing students' knowledge and use of mathematics is more nuanced, and more dependent on the interaction of humans with and in mathematical activity than was accounted for in the "mathematical complexity" approach used in the 2009 to 2017 frameworks.
Mathematical Practices (NEW)	● A new chapter, Chapter 3 – Mathematical Practices, has been written describing and illustrating the assessment of five mathematical practices through which students engage in knowing and doing mathematics. This chapter replaces the previous chapter on Mathematical Complexity. ● Distribution of items for each mathematical practice were developed.	● Since the 1990s, the field of mathematics education has been increasing focus on mathematical processes and the interacting social and mental activities of knowing and doing mathematics. This chapter reflects the field's attention to mathematical activity by describing five mathematical practices. These are assessable aspects of activity at work across mathematics content when students do mathematics. ● Most NAEP Mathematics Assessment items will feature at least one of the five mathematical practices (75 to 80 percent). The range of 75 to 80 percent allows flexibility in assessment and item development across grades 4, 8 and 12 while also ensuring that the majority of the assessment is designed to capture information on student knowledge while engaging in mathematical practices. The balance of items (20 to 25 percent), will assess

		knowledge of content without calling on a particular mathematical practice (e.g., procedural or computational skill).
Item Formats and Assessment Design	● Two chapters in the previous framework (Item Formats and Design of Test and Items) were merged into a single chapter, Overview of the Assessment Design, and updated extensively. ● A new task type, scenario-based tasks, was introduced. Distribution of assessment time for each task type was revised to reflect the introduction of scenario-based tasks and attention to mathematical practices.	● The combination of chapters on assessment and item design allowed addressing interrelationships among: (1) the new digital format of NAEP administration, (2) developments in technology for assessment, including scenario-based tasks, and (3) specifying relationships across task types and assessment time. ● There is a need to ground the NAEP assessment in relevant tasks and familiar contexts to provide a better measure of student content knowledge and mathematical practices (Eklöf, 2010). With the addition of scenario-based tasks the NAEP Mathematics Assessment continues to provide greater access to all students, diversifies the ways in which student achievement can be recognized and measured, and more robustly assesses both what students know and what they can do.
Calculator Policy	Continuing the policy established for the 2017 digital administration of NAEP, students will have access to a calculator emulator in blocks of items designated as “calculator blocks”: four-function for grade 4, scientific for grade 8. The one change in 2025 will be that the grade 12 calculator will include a graphing emulator.	High school students typically use graphing calculators or online emulators and not scientific calculators (Crowe & Ma, 2010).

Item Types	Chapter 4 includes updates to reflect current and future digital platform use and the new item option of scenario-based tasks.	To better assess the diversity of ways of doing mathematics, technology available now and in the near future allows scenario-based items. Such items can be used to assess aspects of mathematical activity that have been difficult (if not impossible) to assess in the past. Building on the work in the last five years to use scenario-based tasks in NAEP Science and NAEP Technology and Engineering Literacy Assessments, Chapter 4 details the ways scenario-based and traditional discrete items can be combined to assess achievement in mathematics content and mathematical practices.
Tools and Manipulatives	Students will continue to have the tools and manipulatives used in the digital administration of the 2017 NAEP Mathematics Assessment. Chapter 4 also explores the potential of behind-the-scenes technology to capture and use process data for assessment; these are data generated by students as they work with digital tools and manipulatives.	The existing digital system tools and mathematics-specific tools have proven worthwhile since the 2017 administration. Additionally, in acknowledgement of the continuing evolution and use of technology in mathematics, Chapter 4 includes examples of other tools (e.g., simulations, dynamic geometry software) that may be common in 2025 and beyond.

Chapter 2

MATHEMATICS CONTENT

The NAEP Mathematics Assessment measures what mathematics students know and are able to do, which involves understanding of particular mathematical ideas (content) and of how to use those ideas in mathematical activity (practices). The content of mathematics can be described by nouns: numbers, data, variables, functions, graphs, geometric figures of various kinds, and the like. In contrast, mathematical practices can be described by verbs: recognize, generalize, deduce, justify and other processes of mathematical reasoning; represent, use, symbolize and other actions involved in applying mathematics; describe, explain, model, and other activities inherent in mathematics being a discipline that is socially constructed by, and communicated among, individuals and societies. This chapter focuses on the mathematics content objectives for the NAEP Mathematics Assessment; Chapter 3 focuses on the practices.

Content Areas

Since its first mathematics assessments in the early 1970s, NAEP has regularly gathered data on students' understanding of five broad areas of mathematics content. These reflect common educational practice in the U.S. While the names of the content areas and some targets for assessment may change from one assessment to the next, NAEP Mathematics content is anchored in these five areas:

- **Number Properties and Operations** (including computation and understanding of number concepts)
- **Measurement** (including use of instruments, application of processes, and concepts of area and volume)
- **Geometry** (including spatial reasoning and applying geometric properties)
- **Data Analysis, Statistics, and Probability** (including graphical displays and statistics)
- **Algebra** (including representations and relationships)

Classification of items into one primary content area is not always clear-cut, but it helps ensure that the indicated mathematical concepts and skills are assessed in a balanced way.

It is worth noting that at grade 12, geometry and measurement are combined as a content area. This reflects the fact that the majority of measurement topics suitable for high school students are geometric in nature. It is also important to note that certain aspects of mathematics occur in all content areas. For example, there is no single objective for computation. Instead, computation is embedded in many content objectives. In this chapter, computation appears in the Number Properties and Operations objectives, which encompass a wide range of concepts about the numeration system and explicitly include a variety of computational skills, ranging from operations with whole numbers to work with decimals, fractions, percents, and real and complex numbers. Computation is also critical in Measurement and Geometry in determining, for example, the perimeter of a rectangle, estimating the height of a building, or finding the hypotenuse of a right triangle. Data analysis often involves computation in calculating a mean, or other statistics describing a collection of numbers, or in calculating probabilities. Solving algebraic equations also frequently involves numerical computation.

To fully understand these objectives and their intent, please note the following: These objectives describe what is to be assessed on NAEP given its operational limitations. They should not be interpreted as a complete description of mathematics that should be taught at these grade levels.

- Some of the grade 12 objectives are marked with an asterisk (*). This denotes objectives that describe mathematics content beyond what is typically taught in a 3-year course of study (the equivalent of 1 year of geometry and 2 years of algebra, with statistics included). Therefore, these objectives will be selected less often than the others for inclusion on the assessments.
- Some objectives in grade 12 are marked with a number sign (# sign). This designates an objective that is an opportunity to assess mathematical literacy (e.g., as a consumer, worker, citizen).
- Although all assessment items will be assigned a primary classification, some items could potentially fall into more than one content area or under more than one objective.
- When the word "or" is used in an objective, it means that an item may assess one or more of the concepts included.
- Further clarification of some objectives along with sample items may be found in *Assessment and Item Specifications.*

Revisions of the 2017 Content Objectives

NAEP mathematics content objectives were evaluated against a range of indicators of educational relevance. These included research on mathematical learning and development, reviews of state standards (e.g., Johnston et al., 2018), national policy such *Guidelines for Assessment and Instruction in Statistics Education* (GAISE, Franklin et al., 2007) and *Guidelines in Assessment and Instruction in Mathematical Modeling Education* (GAIMME), and work on international assessments (e.g., Programme for International Student Assessment, OECD, 2019; Trends in International Mathematics and Science Study, NCES, 2019).

Though overlapping, these sources were not in complete agreement regarding the mathematics students need to know and be able to do. Using this range of sources results in a set of objectives that cannot and will not be representative of what every child in the U.S. is taught in a given grade. At the same time, it is tightly linked to nationally acknowledged aspirations for the mathematics all students in the U.S. should have an opportunity to learn.

Revisions were motivated by several considerations, including precision and accuracy of the language used to describe an objective, developmental appropriateness of objectives at a particular grade level based on current research, and shifts in content emphases since the last framework update. The updates to the objectives include:

- At grade 4, some content in the 2017 NAEP objectives was not regularly part of schooling until grade 6 (Daro et al., 2015; Hughes et al., 2013; Johnston et al., 2018). To address this, six objectives were removed at grade 4 where grade 8 objectives were similar and more appropriately timed to assess students on mathematics they would have had a chance to learn. Also, attention in early grades to equation as an equivalence between two values led to the addition of one objective in grade 4 Algebra. Research suggested that no other objective was absent from NAEP that was commonly assessed in states (Johnston et al., 2018).

- At grade 8, to respond to shifts in expectations about understanding and use of rates and recognition of pattern evident from the research on state and national standards, three objectives were edited, one was deleted in Number and Operations, and one was added in Algebra (Johnston, et al., 2018).
- At grade 12, descriptions of objectives were edited and, in Measurement, one existing objective was identified as beyond what is commonly taught in grade 12 and (*) added; in Geometry one new advanced (*) objective was added to reflect attention to graphical as well as symbolic knowledge of trigonometric functions.
- With the introduction of Practices (see Chapter 3), the Mathematical Reasoning subtopics in Number and Operations, Geometry, Data Analysis, Statistics, and Probability, and Algebra were no longer needed. To preserve attention to the content that was uniquely present in some of the Mathematical Reasoning objectives, objectives in other subtopics were edited (e.g., Number and Operations subtopic 3e in grades 4 and 8 was "Interpret…" and is now "Interpret, explain, or justify…" - for full details on all changes, see the *Assessment and Item Specifications* document).
- Distribution of items for each content area at grades 4 and 12 remains the same. In grade 8, the proportion of items in Data Analysis, Statistics, and Probability was increased 5% (to 20%) and for Algebra decreased by 5% (to 25%). This reflects the increase in attention to data-driven mathematics skills in late elementary and middle grades. Research across state standards did not indicate any adjustment in distributions was needed for grades 4 or 12 (Johnston et al., 2018).

The last several decades have seen a refocusing on the use of mathematics represented by the GAISE and GAIMME guidelines and worldwide attention to mathematical literacy: the application of numerical, spatial, or symbolic mathematical information to situations in a person's life as a consumer, worker, or citizen. At Grades 4 and 8, many instances of mathematical literacy are found in the standard content taught in schools, have been in previous NAEP frameworks, and remain in the objectives enumerated here. In high schools, historically, these topics have had less attention in previous NAEP frameworks. Here, throughout grade 12, objectives that provide opportunities for assessment in the realm of mathematical literacy are identified by the symbol (#). These objectives may also provide opportunities for items that do not fall in this realm. The goal of this identification is to support exploration of NAEP reporting on mathematical literacy.

Item Distribution

The distribution of items among the various mathematics content areas is a critical feature of the assessment design because it reflects the relative importance given to each area in the assessment. As has been the case with past NAEP assessments, the categories have different emphases at each grade. Exhibit 2.1 (next page) provides the recommended balance of items in the assessment by content area for each grade (4, 8, and 12). The percentages refer to the proportion of items, not the amount of testing time.

Exhibit 2.1. Percentage Distribution of Items by Grade and Content Area

Content Area	Grade 4	Grade 8	Grade 12
Number Properties and Operations	40	20	10
Measurement	20	15	30
Geometry	15	20	
Data Analysis, Statistics, and Probability	10	20	25
Algebra	15	25	35

NAEP Mathematics Objectives Organization
Mathematical ideas in different content areas are often interconnected. Organizing the framework by content areas has the potential for obscuring these connections and leading to fragmentation. However, the intent is that the objectives and the items built on them will, in many cases, cross content area boundaries.

To provide clarity and specificity in grade level objectives, the framework matrix (Exhibits 2.2, 2.3, 2.4, 2.5, and 2.6) depicts the particular objectives appropriate for assessment under each subtopic. For example, within the Number Properties and Operations subtopic of Number Sense, specific objectives are listed for assessment at grades 4, 8, and 12. The same objective at different grade levels indicates a developmental sequence for that concept or skill. An empty cell in the matrix conveys that a particular objective is not appropriate or not deemed as important as other areas for assessment at that grade level.

Mathematics Areas

Number Properties and Operations

Numbers are the main tools for describing the world quantitatively. With whole numbers, students can count collections of discrete objects of any type. Students can also use numbers to describe fractional parts, to describe continuous quantities such as length, area, volume, weight, and time, and even to describe more complicated derived quantities such as rates of speed, density, inflation, interest, and so on. Thanks to Cartesian coordinates, ordered pairs of numbers describe points in a plane and ordered triples of numbers can label points in space. Numbers allow precise communication about anything that can be counted, measured, or located in space.

Numbers are not simply labels for quantities; they form systems with their own internal structure. For instance, at times problems can be more easily solved by adding up (e.g., 100 – 98 can be thought of as "98 plus what takes us to 100?"). Multiplication is connected to the idea of repeated addition just as division is connected to the idea of repeated subtraction and the relationship between multiplication and division can be used to simplify computation (e.g., instead of multiplying a number by 25, a number can be multiplied by 100 and then divided by 4, perhaps by halving and halving again). Arithmetic operations (addition, subtraction, multiplication, and division) and the relationships among them help students model basic real-world operations. For example, joining two collections or laying two lengths end to end can be described by addition while comparing two collections can be described by subtraction and the concept of rate depends on division. Multiplication and division of whole numbers lead to the beginnings of number theory, including concepts of factorization, remainder, and prime number. Another basic structure of real numbers is ordering, as in which is greater and which is lesser. Attention to the relative size of quantities provides a basis for making sensible estimates.

Ancient cultures around the world had names for numbers and ways of doing arithmetic. The accessibility and usefulness of arithmetic today is greatly enhanced by the worldwide use of the Hindu-Arabic decimal place value system. In its full development, this remarkable system includes finite and infinite decimals that allow approximating any real number as closely as desired. In fact, all the basic algebraic operations are implicitly used in writing decimal numbers. Decimal notation permits arithmetic by means of routine algorithms, it makes size comparisons straightforward, and estimation simple.

Comfort in dealing with numbers effectively is called *number sense*. It includes intuition about what numbers mean; understanding the ways to represent numbers symbolically (including facility with converting between different representations); ability to calculate, either exactly or approximately, and by several methods (e.g., mentally, with paper and pencil, or with calculator, as appropriate); and ability in estimation. Skill in working with proportions (including percent) is another important part of number sense.

Number sense is a major expectation of the NAEP Mathematics Assessment. In grade 4, students are expected to have a solid grasp of whole numbers as represented by the decimal system and to begin understanding fractions. By grade 8, they should be comfortable with rational numbers, represented either as decimal fractions (including percentages) or as common fractions, and should be able to use them to solve problems involving proportionality and rates. At this level, numbers should also begin to coalesce with geometry by extending students' understanding of the number line. This concept should be connected with the idea of approximation and the use of scientific notation. Grade 8 students should also have some acquaintance with naturally occurring irrational numbers, such as square roots and pi. By grade 12, students should be comfortable dealing with all types of real numbers and various representations, for example, as powers or logarithms. Students in grade 12 should be familiar with complex numbers and be able to establish the validity of numerical properties using mathematical arguments.

Exhibit 2.2. Number Properties and Operations

1) Number sense		
Grade 4	**Grade 8**	**Grade 12**
a) Identify place value and actual value of digits in whole numbers.	a) Use place value to model and describe integers and decimals.	
b) Represent numbers using models such as base 10 representations, number lines, and two-dimensional models.	b) Model or describe rational numbers or numerical relationships using number lines and diagrams.	
c) Compose or decompose whole quantities by place value (e.g., write whole numbers in expanded notation using place value: 342 = 300 + 40 + 2).		
d) Write or rename whole numbers (e.g., 10: 5 + 5, 12 – 2, 2 × 5).	d) Write or rename rational numbers.	# d) Represent, interpret, or compare expressions for real numbers, including expressions using exponents and logarithms.
e) Connect model, number word, or number using various models and representations for whole numbers, fractions, and decimals.	e) Recognize, translate or apply multiple representations of rational numbers (fractions, decimals, and percents) in meaningful contexts.	
	f) Express or interpret large numbers using scientific notation from real-life contexts.	# f) Represent or interpret expressions involving very large or very small numbers in scientific notation.
	g) Find or model absolute value or apply to problem situations.	g) Represent, interpret, or compare expressions or problem situations involving absolute values.
h) Recognize and generate simple equivalent (equal) fractions and visually explain why they are equivalent.	h) Order or compare rational numbers (fractions, decimals, percents, or integers) using various models and represent-tations (e.g., number line).	
i) Order or compare whole numbers, decimals, or fractions.	i) Order or compare rational numbers including very large and small integers, and decimals and fractions close to zero.	i) Order or compare rational or irrational numbers, including very large and very small real numbers.

* Objectives that describe mathematics content beyond that typically taught in a standard 3-year course of study (the equivalent of 1 year of geometry and 2 years of algebra with statistics).
# Objectives that provide opportunities for questions in the realm of mathematical literacy.

Exhibit 2.2 (continued). Number Properties and Operations

2) Estimation		
Grade 4	**Grade 8**	**Grade 12**
a) Use benchmarks (well-known numbers used as meaningful points for comparison) for whole numbers, decimals, or fractions in contexts (e.g., ½ and .5 may be used as benchmarks for fractions and decimals between 0 and 1.00).	a) Establish or apply benchmarks for rational numbers and common irrational numbers (e.g., π) in contexts.	
b) Make estimates appropriate to a given situation with whole numbers, fractions, or decimals by: • Knowing when to estimate, • Selecting the appropriate type of estimate, including overestimate, underestimate, and range of estimate, or • Selecting the appropriate method of estimation (e.g., rounding).	b) Make estimates appropriate to a given situation by: • Identifying when estimation is appropriate, • Determining the level of accuracy needed, • Selecting the appropriate method of estimation, or • Analyzing the effect of an estimation method on the accuracy of results.	# b) Identify situations where estimation is appropriate, determine the needed degree of accuracy, and analyze* the effect of the estimation method on the accuracy of results.
c) Verify and defend solutions or determine the reasonableness of results in meaningful contexts.	c) Verify solutions or determine the reasonableness of results in a variety of situations, including calculator and computer results.	# c) Verify solutions or determine the reasonableness of results in a variety of situations.
	d) Estimate square or cube roots of numbers less than 150 between two whole numbers.	d) Estimate square or cube roots of numbers less than 1,000 between two whole numbers.

* Objectives that describe mathematics content beyond that typically taught in a standard 3-year course of study (the equivalent of 1 year of geometry and 2 years of algebra with statistics).
# Objectives that provide opportunities for questions in the realm of mathematical literacy.

Exhibit 2.2 (continued). Number Properties and Operations

3) Number operations		
Grade 4	**Grade 8**	**Grade 12**
a) Add and subtract: • Whole numbers, or • Fractions with like denominators, or • Decimals through hundredths.	a) Perform computations with rational numbers.	a) Find integral or simple fractional powers of real numbers.
b) Multiply whole numbers: • No larger than two digit by two digit with paper and pencil computation, or • Larger numbers with use of calculator.		b) Perform arithmetic operations with real numbers, including common irrational numbers.
c) Divide whole numbers: • Up to three digits by one digit with paper and pencil computation, or • Up to five digits by two digits with use of calculator.		c) Perform arithmetic operations with expressions involving absolute value.
	d) Describe the effect of operations on size, including the effect of attempts to multiply or divide a rational number by: • Zero, or • A number less than zero, or • A number between zero and one, or • One, or • A number greater than one.	d) Describe the effect of multiplying and dividing by numbers including the effect of attempts to multiply or divide a real number by: • Zero, or • A number less than zero, or • A number between zero and one, or • One, or • A number greater than one.
e) Interpret, explain, or justify whole number operations and explain the relationships between them.	e) Interpret, explain, or justify rational number operations and explain the relationships between them.	e) *Analyze or interpret a proof by mathematical induction of a simple numerical relationship.
f) Solve application problems involving numbers and operations.	f) Solve application problems involving rational numbers and operations using exact answers or estimates as appropriate.	# f) Solve application problems involving numbers, including rational and common irrationals.

* Objectives that describe mathematics content beyond that typically taught in a standard 3-year course of study (the equivalent of 1 year of geometry and 2 years of algebra with statistics).
# Objectives that provide opportunities for questions in the realm of mathematical literacy.

Exhibit 2.2 (continued). Number Properties and Operations

4) Ratios and proportional reasoning		
Grade 4	**Grade 8**	**Grade 12**
a) Use simple ratios to describe problem situations.	a) Use ratios to describe problem situations.	
	b) Use fractions to represent and express ratios and proportions.	
	c) Use proportional reasoning to model and solve problems (including rates and scaling).	# c) Use proportions to solve problems (including rates of change and per capita problems).
	d) Solve problems involving percentages (including percent increase and decrease, interest rates, tax, discount, tips, or part/whole relationships).	# d) Solve multistep problems involving percentages, including compound percentages.

5) Properties of number and operations		
Grade 4	**Grade 8**	**Grade 12**
a) Identify odd and even numbers		
b) Identify factors of whole numbers	b) Recognize, find, or use factors, multiples, or prime factorization.	
	c) Recognize or use prime and composite numbers to solve problems.	c) Solve problems using factors, multiples, or prime factorization.
	d) Use divisibility or remainders in problem settings.	# d) Use divisibility or remainders in problem settings.
e) Apply basic properties of operations.	e) Apply basic properties of operations.	e) Apply basic properties of operations, including conventions about the order of operations.
		f) Recognize properties of the number system (whole numbers, integers, rational numbers, real numbers, and complex numbers) and how they are related to each other, and identify examples of each type of number.

Objectives that provide opportunities for questions in the realm of mathematical literacy.

Measurement

Measuring is the process by which numbers are assigned to describe the world quantitatively. This process involves selecting the attribute of the object or event to be measured, comparing this attribute to a unit, and reporting the number of units. For example, in measuring a banner, one may select the attribute of length and the inch as the unit for the comparison. In comparing lengths to the nearest inch, it may be that a length is *about* 42 inches. If considering only the domain of whole numbers, one would report that the banner is 42 inches long. However, since length is a continuous attribute, in the domain of rational numbers the length of the banner would be reported as $41^{3}/_{16}$ inches (to the nearest 16th of the inch).

The connection between measuring and number makes measurement a vital part of mathematical learning. Measurement is an important setting for negative and irrational numbers as well as positive numbers. Measurement models are often used when students are learning about number and operations. For example, area and volume models can help students understand multiplication and its properties. Length models, especially the number line, can help students understand ordering and rounding numbers. Measurement also has a strong connection to other areas of school mathematics and to other subjects. Problems in algebra are often drawn from measurement situations and functions often relate measures to each other. Geometry regularly focuses on measurement aspects of geometric figures. Probability and statistics provide ways to measure chance and to compare sets of data. The measurement of time, values of goods and services, physical properties of objects, distances, and various kinds of rates exemplify the importance of measurement in everyday activities.

In this framework, attributes such as capacity, weight, mass, time, and temperature are included, as are the geometric attributes of length, area, and volume. Many of these attributes appear in grade 4, where the emphasis is on length, including perimeter, distance, and height. More emphasis is placed on areas and angle measures in grade 8. By grade 12, measurement in everyday life – as well as in the study of volumes and rates constructed from other attributes, such as speed – are emphasized.

The NAEP Mathematics Assessment includes nonstandard, customary, and metric units. At grade 4, common customary units such as inch, quart, pound, and hour; and common metric units such as centimeter, liter, and gram are emphasized. Grades 8 and 12 include the use of both square and cubic units for measuring area, surface area, and volume, degrees for measuring angles, and constructed units such as miles per hour. Converting from one unit in a system to another, such as from minutes to hours, is an important aspect of measurement included in problem situations. Understanding and using the many conversions available is an important skill. There are a limited number of common, everyday equivalencies that students are expected to know (see *Assessment and Item Specifications* for more detail).

Items classified in this content area depend on some knowledge of measurement. For example, an item that asks for the difference between a 3-inch and a 1¾ -inch line segment is a number item, whereas an item comparing a 2-foot segment with an 8-inch line segment is a measurement item. In many secondary schools, measurement becomes an integral part of geometry; this is reflected in the proportion of items recommended for these two areas (see Exhibit 2.1).

Exhibit 2.3. Measurement

1) Measuring physical attributes		
Grade 4	**Grade 8**	**Grade 12**
a) Identify the attribute that is appropriate to measure in a given situation.		
b) Compare objects with respect to a given attribute, such as length, area, volume, time, or temperature.	b) Compare objects with respect to length, area, volume, angle measurement, weight, or mass.	# b) Determine the effect of proportions and scaling on length, area, and volume.
c) Estimate the size of an object with respect to a given measurement attribute (e.g., length, perimeter, or area using a grid).	c) Estimate the size of an object with respect to a given measurement attribute (e.g., area).	# c) Estimate or compare perimeters or areas of two-dimensional geometric figures.
		d) Solve problems of angle measure, including those involving triangles or other polygons or parallel lines cut by a transversal.
e) Select or use appropriate measurement instruments such as ruler, meter stick, clock, thermometer, or other scaled instruments.	e) Select or use appropriate measurement instrument to determine or create a given length, area, volume, angle, weight, or mass.	
f) Solve problems involving perimeter of plane figures.	f) Solve mathematical or real-world problems involving perimeter or area of plane figures such as triangles, rectangles, circles, or composite figures.	f) Solve problems involving perimeter or area of plane figures such as polygons, circles, or composite figures.
g) Solve problems involving area of squares and rectangles.		
	h) Solve problems involving volume or surface area of rectangular solids, cylinders, prisms, or composite shapes.	h) Solve problems by determining, estimating, or comparing volumes or surface areas of three-dimensional figures.
	i) Solve problems involving rates such as speed or ratios such as population density.	# i) Solve problems involving rates and ratios such as speed, density, population density, or flow rates.

Objectives that provide opportunities for questions in the realm of mathematical literacy.

Exhibit 2.3 (continued). Measurement

2) Systems of measurement		
Grade 4	**Grade 8**	**Grade 12**
a) Select or use an appropriate type of unit for the attribute being measured such as length, time, or temperature.	a) Select or use an appropriate type of unit for the attribute being measured such as length, area, angle, time, or volume.	a) Recognize that geometric measurements (length, area, perimeter, volume) depend on choice of a unit, and apply such units in expressions, equations, and problem solutions.
b) Solve problems involving conversions within the same measurement system such as conversions involving inches and feet or hours and minutes.	b) Solve problems involving conversions within the same measurement system such as conversions involving square inches and square feet.	# b) Solve problems involving conversions within or between measurement systems, given the relationship between the units.
	c) Estimate the measure of an object in one system given the measure of that object in another system and the approximate conversion factor. For example: • Distance: 1 kilometer is approximately .6 of a mile. • Money: U.S. dollars to Canadian dollars. • Temperature: Fahrenheit to Celsius.	
d) Determine appropriate unit of measurement in problem situations involving such attributes as length, time, capacity, or weight.	d) Determine appropriate unit of measurement in problem situations involving such attributes as length, area, or volume.	# d) Understand that numerical values associated with measurements of physical quantities are approximate, subject to variation, and must be assigned units of measurement.
e) Determine situations in which a highly accurate measurement is important.	e) Determine appropriate accuracy of measurement in problem situations (e.g., the accuracy of each of several lengths needed to obtain a specified accuracy of a total length) and find the measure to that degree of accuracy.	e) # Determine appropriate accuracy of measurement in problem situations (e.g., the accuracy of measurement of the dimensions to obtain a specified accuracy of area) and find the measure to that degree of accuracy.
	f) Construct or solve problems (e.g., floor area of a room) involving scale drawings.	f) # Construct or solve problems involving scale drawings.

Objectives that provide opportunities for questions in the realm of mathematical literacy.

Exhibit 2.3 (continued). Measurement

3) Measurement in triangles		
Grade 4	**Grade 8**	**Grade 12**
	a) Solve problems involving indirect measurement such as finding the height of a building by comparing its shadow with the height and shadow of a known object.	# a) Solve problems involving indirect measurement.
		b) Solve problems using the fact that trigonometric ratios (sine, cosine, and tangent) stay constant in similar triangles.
		c) Use the definitions of sine, cosine, and tangent as ratios of sides in a right triangle to solve problems about length of sides and measure of angles.
		d) * Interpret and use the identity $\sin^2\theta + \cos^2\theta = 1$ for angles θ between 0° and 90°; recognize this identity as a special representation of the Pythagorean theorem.
		e) * Determine the radian measure of an angle and explain how radian measurement is related to a circle of radius 1.
		f) * Use trigonometric formulas such as addition and double angle formulas.
		g) * Use the law of cosines and the law of sines to find unknown sides and angles of a triangle.
		h) * Interpret the graphs of the sine, cosine, and tangent functions with respect to periodicity and values of these functions for multiples of $\pi/6$ and $\pi/4$.

* Objectives that describe mathematics content beyond that typically taught in a standard 3-year course of study (the equivalent of 1 year of geometry and 2 years of algebra with statistics).
# Objectives that provide opportunities for questions in the realm of mathematical literacy.

Geometry

Geometry began thousands of years ago in many lands as sets of practical rules related to describing and predicting locations of astronomical objects, for calculating land areas, and for building structures. More than 2200 years ago, the Greek mathematician Euclid organized the geometry known at that time into a coherent collection of results, all deduced using logic from a small number of postulates assumed to be true. Euclid's work was fundamental in establishing mathematical truth as dependent on valid deductive reasoning rather than reliant on educated guesses from a number of specific examples. The theorems obtained by deduction in Euclid's work remain fundamental to the study of geometry and for this reason the geometry studied in school is called Euclidean geometry.

The fundamental concepts of Euclidean geometry are congruence, similarity, and symmetry. By grade 4, students are expected to be familiar with a library of simple figures and their attributes, both in the plane (lines, circles, triangles, rectangles, and squares) and in space (cubes, spheres, and cylinders).

In middle school, understanding of these shapes deepens, with study of cross-sections of solids and the beginnings of an analytical understanding of properties of plane figures, especially parallelism, perpendicularity, and angle relations in polygons. Reflections, translations, and rotations (mathematical models of the physical phenomena of reflecting, sliding, and turning) are introduced as functions that map a figure onto a congruent image because they preserve distance. Dilatations (expansions and contractions) map figures onto similar images. By grade 8, properties of congruent and similar figures involve angle measures and lengths, so geometry becomes more and more mixed with measurement. Placing figures on a coordinate plane provides the beginnings of the connections among algebra, geometry, and analytic geometry.

In secondary school, the content of plane geometry is logically ordered and students are expected to make, test, and validate conjectures. Students see that most of the commonly-studied plane figures – the triangles (scalene, isosceles, equilateral) and quadrilaterals (parallelogram, rectangle, rhombus, square) – possess reflection or rotation symmetry, or both, and can use triangle congruence and similarity theorems as well as symmetry to establish properties of figures. By grade 12, students may also gain insight into systematic structure, such as the classification of distance-preserving transformations of the plane (that is, isometries and congruence transformations as reflections, rotations, translations, or glide reflections), and what happens when two or more isometries are performed in succession (composition). In analytic geometry, the key areas of geometry and algebra merge into a powerful tool that provides a basis for calculus and much of applied mathematics.

Exhibit 2.4. Geometry

1) Dimension and shape		
Grade 4	**Grade 8**	**Grade 12**
	a) Draw or describe a path of shortest length between points to solve problems in context.	
b) Identify or describe (informally) real-world objects using simple plane figures (e.g., triangles, rectangles, squares, and circles) and simple solid figures (e.g., cubes, spheres, and cylinders).	b) Identify a geometric object given a written description of its properties.	
c) Identify, measure, or draw angles and other geometric figures in the plane.	c) Identify, define, or describe geometric shapes in the plane and in three-dimensional space given a visual representation.	c) Give precise mathematical descriptions or definitions of geometric shapes in the plane and in three-dimensional space.
	d) Draw or sketch from a written description polygons, circles, or semicircles.	d) Draw or sketch from a written description plane figures and planar images of three-dimensional figures.
	e) Represent or describe a three-dimensional situation in a two-dimensional drawing from different views.	# e) Use two-dimensional representations of three-dimensional objects to visualize and solve problems.
f) Describe or distinguish among attributes of two- and three-dimensional shapes.	f) Demonstrate an understanding about two- and three-dimensional shapes in the world through identifying, drawing, modeling, building, or taking apart.	f) Analyze properties of three-dimensional figures including prisms, pyramids, cylinders, cones, spheres and hemispheres.

2) Transformation of figures and preservation of properties		
Grade 4	**Grade 8**	**Grade 12**
a) Identify whether a figure is symmetrical or draw lines of symmetry.	a) Identify lines of symmetry in plane figures or recognize and classify types of symmetries of plane figures.	a) Recognize or identify types of symmetries (e.g., translation, reflection, rotation) of two- and three-dimensional figures.
		b) Give or recognize the precise mathematical relationship (e.g., congruence, similarity, orientation) between a figure and its image under a transformation.

Objectives that provide opportunities for questions in the realm of mathematical literacy.

Exhibit 2.4 (continued). Geometry

2) Transformation of figures and preservation of properties (continued)		
Grade 4	**Grade 8**	**Grade 12**
c) Identify the images resulting from flips (reflections), slides (translations), or turns (rotations).	c) Recognize or informally describe the effect of a transformation (reflection, rotation, translation, expansion, or contraction) on two-dimensional figures.	c) Perform or describe the effect of a single transformation (reflection, rotation, translation, or dilation) on two- or three-dimensional geometric figures.
d) Recognize which attributes (such as shape and area) change or do not change when plane figures are cut up or rearranged.	d) Predict results of combining, subdividing, and changing shapes of plane figures and solids (e.g., paper folding, tiling, cutting up and rearranging pieces).	d) Identify transformations, combinations, or subdivisions of shapes that preserve the area of two-dimensional figures or the volume of three-dimensional figures.
	e) Justify relationships of congruence and similarity and apply these relationships using scaling and proportional reasoning.	e) Justify relationships of congruence and similarity and apply these relationships using scaling and proportional reasoning.
	f) Apply the relationships among angle measures, lengths, and perimeters among similar figures.	
		g) Perform or describe the effects of successive (composites of) transformations.

3) Relationships between geometric figures		
Grade 4	**Grade 8**	**Grade 12**
a) Analyze or describe patterns of geometric figures by increasing number of sides, changing size or orientation (e.g., polygons with more and more sides).		
b) Assemble simple plane shapes to construct a given shape.	b) Apply geometric properties and relationships in solving simple problems in two and three dimensions.	b) Apply geometric properties and relationships to solve problems in two and three dimensions.

Exhibit 2.4 (continued). Geometry

3) Relationships between geometric figures (continued)		
Grade 4	**Grade 8**	**Grade 12**
c) Recognize two-dimensional faces of three-dimensional shapes.	c) Represent problem situations with simple geometric models to solve mathematical or real-world problems.	# c) Represent problem situations with geometric models to solve mathematical or real-world problems.
	d) Use the Pythagorean theorem to solve problems.	# d) Use the Pythagorean theorem to solve problems in two- or three-dimensional situations.
		e) Recall and interpret or use definitions and basic properties of congruent and similar triangles, circles, quadrilaterals, polygons, parallel, perpendicular and intersecting lines, and associated angle relationships (e.g., in solving problems or creating proofs).
f) Describe and compare properties of simple and compound figures composed of triangles, squares, and rectangles.	f) Describe or analyze simple properties of, or relationships between, triangles, quadrilaterals, and other polygonal plane figures.	f) Analyze properties or relationships of triangles, quadrilaterals, and other polygonal plane figures.
	g) Describe or analyze properties and relationships of parallel or intersecting lines.	g) Analyze properties and relationships of parallel, perpendicular, or intersecting lines including the angle relationships that arise in these cases.
	h) Make and test a geometric conjecture about triangles, quadrilaterals, or other polygons.	h) Make, test, and validate geometric conjectures using a variety of methods including deductive reasoning and counterexamples
		i) * Analyze properties of circles and the intersections of lines and circles (inscribed angles, central angles, tangents, secants, and chords).

* Objectives that describe mathematics content beyond that typically taught in a standard 3-year course of study (the equivalent of 1 year of geometry and 2 years of algebra with statistics).
# Objectives that provide opportunities for questions in the realm of mathematical literacy.

Exhibit 2.4 (continued). Geometry

4) Position, direction, and coordinate geometry		
Grade 4	**Grade 8**	**Grade 12**
a) Describe relative positions of points and lines using the geometric ideas of parallelism or perpendicularity.	a) Describe relative positions of points and lines using the geometric ideas of midpoint, points on common line through a common point, parallelism, or perpendicularity.	a) Solve problems involving the coordinate plane such as the distance between two points, the midpoint of a segment, or slopes of perpendicular or parallel lines.
	b) Describe the intersection of two or more geometric figures in the plane (e.g., intersection of a circle and a line).	b) Describe the intersections of lines in the plane and in space, intersections of a line and a plane, or of two planes in space.
	c) Visualize or describe the cross section of a solid.	# c) Describe or identify conic sections and other cross sections of solids.
d) Draw geometric figures with vertices at points on a coordinate grid.	d) Represent geometric figures using rectangular coordinates on a plane.	d) Represent two-dimensional figures algebraically using coordinates and/or equations.
		e) * Use vectors to represent velocity and direction; multiply a vector by a scalar and add vectors both algebraically and graphically.
		f) Find an equation of a circle given its center and radius and, given an equation of a circle, find its center and radius.
		g) * Graph or determine equations for images of lines, circles, parabolas, and other curves under translations and reflections in the coordinate plane.
		h) * Represent situations and solve problems involving polar coordinates.

* Objectives that describe mathematics content beyond that typically taught in a standard 3-year course of study (the equivalent of 1 year of geometry and 2 years of algebra with statistics).
# Objectives that provide opportunities for questions in the realm of mathematical literacy.

Data Analysis, Statistics, and Probability

Data analysis and statistics covers the entire process of collecting, organizing, summarizing, and interpreting data. This is the heart of statistics and is in evidence whenever quantitative information is used to determine a course of action. To emphasize the spirit of statistical thinking, data analysis begins with a question to be answered, not with a set of data. Data should be collected only with a specific question (or questions) in mind and only after a plan (usually called a *design*) for collecting data relevant to the question is thought out. Beginning at an early age, students should grasp the fundamental principle that exploratory data analysis done by examining an existing data set is far different from the scientific method of collecting data to verify or refute a well-posed question. Patterns can be found in almost any data set if one looks hard enough; however, patterns discovered in this way are often meaningless from the point of view of statistical inference.

A probability is a measure of uncertainty. This measure may be an assumption, as when one says that the probability of an evenly balanced coin landing head-side up is one-half (even if that coin has never been tossed) or it may be determined in some way from past experience, as when forecasters say the probability of rain tomorrow is 40 percent. Statistical analysis often involves studying whether assumptions about probability match observed relative frequencies. For instance, if a coin tossed 100 times actually turned up heads 80 times, one would suspect that the probability of heads for that coin is not one-half and that the coin is not balanced. Under random sampling, patterns for outcomes of designed studies can be anticipated and used as the basis for making decisions. The whole probability distribution of all possible outcomes is important in most statistical decision-making because the key is to decide whether or not a particular observed outcome is unusual (located in a tail of the probability distribution) or not. For example, four as a grade-point average is unusually high among most student groups, four as the pound weight of a human baby is unusually low, and four as the number of floors in a building is not unusual in either direction.

By grade 4, students are expected to apply their understanding of number and quantity to pose questions that can be answered by collecting appropriate data. They also are expected to organize data in a table or a plot and summarize the essential features of center, spread, and shape, both verbally and with simple summary statistics, such as median and range. Simple comparisons can be made between two related data sets but more formal inference based on randomness come later. The basic concept of chance and statistical reasoning can be built into meaningful contexts, such as "If I draw two names from among those of the students in the room, am I likely to get two people wearing black shoes?" Such problems can be addressed through simulation.

Building on the same definition of data analysis and the same principles of describing data distributions through center, spread, and shape, grade 8 students are expected to be able to use a wider variety of organizing and summarizing techniques. They can identify and construct a statistical question, one that needs data in order to be addressed. They can also begin to analyze statistical claims through designed surveys and experiments that involve randomization, with simulation being the main tool for making statistical inferences. By grade 8, students are expected to begin to use more formal terminology related to probability and data analysis. They can identify associations between two numerical variables in scatterplots, as well as the relative strength of those associations.

Grade 12 students are expected to use a wide variety of statistical techniques for all phases of the data analysis process, including a more formal understanding of statistical inference (still with simulation as the main inferential analysis tool). Students can pose their own statistical questions given a problem situation or context involving data. In addition to comparing univariate data sets, students at this level can recognize and describe possible associations between two variables by looking at two-way tables for categorical variables or scatterplots for measurement variables. Association between variables is related to the concepts of independence and dependence of events and an understanding of these ideas requires knowledge of conditional probability. By grade 12, students should be able to use statistical models (linear and nonlinear equations) to describe possible associations between measurement variables and should be familiar with techniques for fitting models to data.

Exhibit 2.5. Data Analysis, Statistics, and Probability

1) Data representation		
Grade 4	**Grade 8**	**Grade 12**
Representations of data are indicated below for each grade level. Objectives in which only a subset of these representations is applicable are indicated in the parenthesis associated with the objective.		
Pictographs, bar graphs, circle graphs, dot plots, tables, and tallies.	Histograms, plots over time, dot plots, scatterplots, box plots, bar graphs, circle graphs, stem and leaf plots, frequency distributions, and tables.	Histograms, plots over time, dot plots, scatterplots, box plots, bar graphs, circle graphs, stem and leaf plots, frequency distributions, and tables, including two-way tables.
a) Read or interpret a single set of data.	a) Read or interpret data, including interpolating or extrapolating from data.	# a) Read or interpret graphical or tabular representations of data.
b) For a given set of data, complete a graph (limits of time make it difficult to construct graphs completely).	b) For a given set of data, complete a graph and then solve a problem using the data in the graph (histograms, plots over time, scatterplots, dot plots, circle graphs, and bar graphs).	# b) For a given set of data, complete a graph and solve a problem using the data in the graph (histograms, scatterplots, dot plots, plots over time).
c) Solve problems by estimating and computing within a single set of data.	c) Solve problems by estimating and computing with data from a single set or across sets of data.	c) Solve problems involving univariate or bivariate data.

Objectives that provide opportunities for questions in the realm of mathematical literacy.

Exhibit 2.5 (continued). Data Analysis, Statistics, and Probability

1) Data representation (continued)		
Grade 4	**Grade 8**	**Grade 12**
	d) Given a graph or a set of data, determine whether information is represented effectively and appropriately (histograms, plots over time, box plots, scatterplots, circle graphs, dot plots, bar graphs).	# d) Given a graphical or tabular representation of a set of data, determine whether information is represented effectively and appropriately.
	e) Compare and contrast the effectiveness of different representations of the same data (e.g., identify misleading uses of data in real-world settings).	# e) Compare and contrast different graphical representations of univariate and bivariate data (e.g., identify misleading uses of data in real-world settings and critique different ways of presenting and using information).
		f) * Organize and display data in a spreadsheet in order to recognize patterns and solve problems.

2) Characteristics of data sets		
Grade 4	**Grade 8**	**Grade 12**
	a) Calculate, use, or interpret mean, median, mode, range or shape.	# a) Calculate, interpret, or use summary statistics for distributions of data including measures of typical value (mean, median), position (quartiles, percentiles), spread (range, interquartile range, variance, and standard deviation) or shape (skew, uniform, uni/bi-modal).
b) Given a set of data or a graph, describe the distribution of data using median, range, mode, or shape.	b) Describe how mean, median, mode, range, or interquartile ranges relate to distribution shape.	b) Recognize how linear transformations of one-variable data affect mean, median, mode, range, interquartile range, and standard deviation.
	c) Identify outliers and determine their effect on mean, median, mode, or range.	# c) Determine the effect of outliers on mean, median, mode, range, interquartile range, or standard deviation.

Objectives that provide opportunities for questions in the realm of mathematical literacy.

Exhibit 2.5 (continued). Data Analysis, Statistics, and Probability

2) Characteristics of data sets (continued)		
Grade 4	**Grade 8**	**Grade 12**
d) Compare two sets of related data.	d) Using appropriate statistical measures, compare two or more data sets describing the same characteristic for two different populations or subsets of the same population.	d) Compare data sets using summary statistics (mean, median, mode, range, interquartile range, shape, or standard deviation) describing the same characteristic for two different populations or subsets of the same population.
	e) Visually choose the line that best fits given a scatterplot and informally explain the meaning of the line. Use the line to make predictions.	e) Approximate a trend line if a linear pattern is apparent in a scatterplot or use a graphing calculator to determine a least-squares regression line and use the line or equation to make predictions.
		# f) Recognize or explain how an argument based on data might confuse correlation with causation.
		g) * Know and interpret the key characteristics of a normal distribution such as shape, center (mean), and spread (standard deviation).
		# h) * Recognize and explain the potential errors that can arise when extrapolating from data.

3) Experiments and samples		
Grade 4	**Grade 8**	**Grade 12**
	a) Given a sample, identify possible sources of bias in sampling.	# a) Identify possible sources of bias in sample survey populations or questions and describe how such bias can be controlled and reduced.
	b) Distinguish between a random and nonrandom sample.	b) Recognize and describe a method to select a simple random sample.

* Objectives that describe mathematics content beyond that typically taught in a standard 3-year course of study (the equivalent of 1 year of geometry and 2 years of algebra with statistics).
# Objectives that provide opportunities for questions in the realm of mathematical literacy.

Exhibit 2.5 (continued). Data Analysis, Statistics, and Probability

3) Experiments and samples (continued)		
Grade 4	**Grade 8**	**Grade 12**
		# c) Draw inferences from samples, such as estimates of proportions in a population, estimates of population means, or decisions about differences in means for two "treatments."
	d) Evaluate the design of an experiment.	d) Identify or evaluate the characteristics of a good survey or of a well-designed experiment.
		e) * Recognize the differences in design and in conclusions between randomized experiments and observational studies.

4) Probability		
Grade 4	**Grade 8**	**Grade 12**
a) Use informal probabilistic thinking to describe chance events (i.e., less likely and more likely, certain and impossible.	a) Analyze a situation that involves probability of an event.	# a) Determine whether two events are independent or dependent.
b) Determine a simple probability from a context that includes a picture.	b) Determine the theoretical probability of simple and compound events in familiar contexts.	# b) Determine the theoretical probability of simple and compound events in familiar or unfamiliar contexts.
	c) Estimate the probability of simple and compound events through experimentation or simulation.	# c) Given the results of an experiment or simulation, estimate the probability of simple or compound events in familiar or unfamiliar contexts.

* Objectives that describe mathematics content beyond that typically taught in a standard 3-year course of study (the equivalent of 1 year of geometry and 2 years of algebra with statistics).
# Objectives that provide opportunities for questions in the realm of mathematical literacy.

Exhibit 2.5 (continued). Data Analysis, Statistics, and Probability

4) Probability (continued)		
Grade 4	**Grade 8**	**Grade 12**
	d) Use theoretical probability to evaluate or predict experimental outcomes.	d) Use theoretical probability to evaluate or predict experimental outcomes.
	e) Determine the sample space for a given situation.	e) Determine the number of ways an event can occur using tree diagrams, formulas for combinations and permutations, or other counting techniques.
	f) Use a sample space to determine the probability of possible outcomes for an event.	
	g) Represent the probability of a given outcome using fractions, decimals, and percents.	
	h) Determine the probability of independent and dependent events. (Dependent events should be limited to a small sample size.)	h) Determine the probability of independent and dependent events.
		i) Determine conditional probability using two-way tables.
	j) Interpret probabilities within a given context.	# j) Interpret and apply probability concepts to practical situations, including odds of success or failure in simple lotteries or games of chance.
		k) * Use the binomial theorem to solve problems.

* Objectives that describe mathematics content beyond that typically taught in a standard 3-year course of study (the equivalent of 1 year of geometry and 2 years of algebra and statistics).
# Objectives that provide opportunities for questions in the realm of mathematical literacy.

Algebra

Algebra began in the use of systematic methods for solving problems and numerical puzzles by mathematicians in the Middle East, South Asia, and China, and made its way to Europe in the late Middle Ages. The modern symbolic notation, with letters to stand for unknowns and constants, was developed in the 16th century. The notation so greatly enhanced the power of the algebraic method that the basic ideas of both analytic geometry and calculus were developed within a century.

The widening use of algebra led to study of its formal structure. Gradually, the "rules of algebra" were distilled into a compact summary of the principles behind algebraic manipulation. In the 19th century, these principles (e.g., commutativity, distributivity) were codified into a deductive system parallel to that of Euclidean geometry. A corresponding line of thought produced a simple but flexible concept of function and also led to the development of set theory as a comprehensive background for mathematics. When taken broadly as including these ideas, algebra reaches from the foundations of mathematics to the frontiers of current research.

The concept of variable – a symbol that can stand for any member of an identified set – has multiple facets (e.g., as an unknown, parameter, varying quantity). In describing arithmetic relationships such as the commutativity of addition, variables are pattern generalizers. In formulas such as $d = rt$ or $c = \sqrt{a^2 + b^2}$, variables stand for quantities that may take on a variety of values. In problem solving, a variable may be an unknown while in the study of functions, independent and dependent variables stand for domain and range values and parameters stand for constants.

By grade 4, students are expected to recognize and extend simple numeric patterns as one foundation for a later understanding of function. They begin to understand the meaning of equality and some of its properties, as well as the idea of an as-yet-unknown quantity as a precursor to the concept of variable.

As students move into grade 8, the ideas of variable, covariation (two or more quantities varying simultaneously), and function become more important. By using variables to describe patterns and solve simple equations, students become familiar with manipulating them. Representations of covariation in tables, verbal descriptions, symbolic descriptions, and graphs can combine to promote a flexible grasp of the idea of function. Linear functions receive special attention. They connect to the ideas of proportionality, ratio, and rate, forming a bridge that will eventually link arithmetic to calculus. Symbolic manipulation in the relatively simple context of linear equations is reinforced by other ways of finding solutions, including graphing by hand or with technology.

By grade 12, students are expected to be skillful at manipulating and interpreting more complex expressions. Nonlinear functions, especially quadratic, power, and exponential functions whose graphs are accessible using graphing technology are used by students to solve real-world problems. Grade 12 students are also expected to be accomplished at translating verbal descriptions of problem situations into symbolic form. The algebraic properties of real numbers should come to be appreciated as a basis for reasoning. Also by grade 12, students should understand expressions involving several variables, systems of linear equations, and solutions to inequalities.

Exhibit 2.6. Algebra

1) Patterns, relations, and functions		
Grade 4	**Grade 8**	**Grade 12**
a) Recognize, describe, or extend numerical and visual patterns.	a) Recognize, describe, or extend numerical and geometric patterns using tables, graphs, words, or symbols.	a) Recognize, describe, or extend numerical patterns, including arithmetic and geometric progressions.
b) Given a pattern or sequence, construct, explain, or justify a rule to generate the terms of the pattern or sequence.	b) Generalize or justify a pattern appearing in a numerical sequence, table, or graph using words or symbols.	b) Express linear and exponential functions in recursive and explicit form given a verbal description, table, or some terms of a sequence.
c) Given a description, extend or find a missing term in a pattern or sequence.	c) Analyze or create patterns, sequences, or linear functions given a rule.	
d) Create a different representation of a pattern or sequence given a verbal description.		
e) Recognize or describe a relationship in which quantities change proportionally.	e) Identify functions as linear or nonlinear or contrast distinguishing properties of functions from tables, graphs, or equations.	e) Identify or analyze distinguishing properties of linear, quadratic, rational, exponential, or *trigonometric functions from tables, graphs, or equations.
	f) Interpret the meaning of slope or intercepts, or determine the rate of change between two points on a graph of a linear function	
		g) Determine whether a relation, given in verbal, symbolic, tabular, or graphical form, is a function.
		h) Recognize and analyze the general forms of linear, quadratic, rational, exponential, or *trigonometric functions.
		i) Determine the domain and range of functions given in various forms and contexts.
		j) * Given a function, determine its inverse if it exists and explain the contextual meaning of the inverse for a given situation.

* Objectives that describe mathematics content beyond that typically taught in a standard 3-year course of study (the equivalent of 1 year of geometry and 2 years of algebra and statistics).

Exhibit 2.6 (continued). Algebra

2) Algebraic representations		
Grade 4	**Grade 8**	**Grade 12**
a) Translate between the different forms of representations (symbolic, numerical, verbal, or pictorial) of whole number relationships (such as from a written description to an equation or from a function table to a written description).	a) Translate between different representations of linear expressions using symbols, graphs, tables, diagrams, or written descriptions.	a) Create and translate between different representations of algebraic expressions, equations, and inequalities (e.g., linear, quadratic, exponential, or *trigonometric) using symbols, graphs, tables, diagrams, or written descriptions.
	b) Analyze or interpret linear relationships expressed in symbols, graphs, tables, diagrams, or written descriptions.	b) Analyze or interpret relationships expressed in symbols, graphs, tables, diagrams (including Venn diagrams), or written descriptions and evaluate the relative advantages or disadvantages of different representations to answer specific questions.
c) Graph or interpret points with whole number or letter coordinates on grids or in the first quadrant of the coordinate plane.	c) Graph or interpret points represented by ordered pairs of numbers on a rectangular coordinate system.	
	d) Solve problems involving coordinate pairs on the rectangular coordinate system.	d) Perform or interpret transformations on the graphs of linear, quadratic, exponential, and *trigonometric functions.
		e) Make inferences or predictions using an algebraic model of a situation.

* Objectives that describe mathematics content beyond that typically taught in a standard 3-year course of study (the equivalent of 1 year of geometry and 2 years of algebra and statistics).
\# Objectives that provide opportunities for questions in the realm of mathematical literacy.

Exhibit 2.6 (continued). Algebra

2) Algebraic representations (continued)		
Grade 4	**Grade 8**	**Grade 12**
	f) Identify or represent functional relationships in meaningful contexts including proportional, linear, and common nonlinear (e.g., compound interest, bacterial growth) in tables, graphs, words, or symbols.	f) Given a real-world situation, determine if a linear, quadratic, rational, exponential, logarithmic, or *trigonometric function fits the situation.
		# g) Solve problems involving exponential growth and decay.
		h) *Analyze properties of exponential, logarithmic, and rational functions.
3) Variables, expressions, and operations		
Grade 4	**Grade 8**	**Grade 12**
a) Use letters and symbols to represent an unknown quantity in a simple mathematical expression.		
b) Express simple mathematical relationships using number sentences.	b) Write algebraic expressions, equations, or inequalities to represent a situation.	b) Write algebraic expressions, equations, or inequalities to represent a situation.
	c) Perform basic operations, using appropriate tools, on linear algebraic expressions (including grouping and order of multiple operations involving basic operations, exponents, roots, simplifying, and expanding).	c) Perform basic operations, using appropriate tools, on algebraic expressions including polynomial and rational expressions.
		d) Write equivalent forms of algebraic expressions, equations, or inequalities to represent and explain mathematical relationships.

* Objectives that describe mathematics content beyond that typically taught in a standard 3-year course of study (the equivalent of 1 year of geometry and 2 years of algebra and statistics).
# Objectives that provide opportunities for questions in the realm of mathematical literacy.

Exhibit 2.6 (continued). Algebra

3) Variables, expressions, and operations (continued)		
Grade 4	**Grade 8**	**Grade 12**
		# e) Evaluate algebraic expressions, including polynomials and rational expressions.
		f) Use function notation to evaluate a function at a specified point in its domain and combine functions by addition, subtraction, multiplication, division, and composition.
		g) * Determine the sum of finite and infinite arithmetic and geometric series.
		h) Use basic properties of exponents and *logarithms to solve problems.

4) Equations and inequalities		
Grade 4	**Grade 8**	**Grade 12**
a) Find the value of the unknown in a whole number sentence (e.g., in an equation or simple inequality like [_] + 3 > 7).	a) Solve linear equations or inequalities (e.g., Solve for x in $ax + b = c$ or $ax + b = cx + d$ or $ax + b > c$).	a) Solve linear, rational, or quadratic equations or inequalities, including those involving absolute value.
b) Interpret "=" as an equivalence between two values and use this interpretation to solve problems.	b) Interpret "=" as an equivalence between two expressions and use this interpretation to solve problems.	b) * Determine the role of hypotheses, logical implications, and conclusions in algebraic arguments about equality and inequality.
c) Verify a conclusion using algebraic properties.	c) Make, validate, and justify conclusions and generalizations about linear relationships.	c) Use algebraic properties to develop a valid mathematical argument.
	d) Analyze situations or solve problems using linear equations and inequalities with rational coefficients symbolically or graphically (e.g., $ax + b = c$ or $ax + b = cx + d$).	# d) Analyze situations, develop mathematical models, or solve problems using linear, quadratic, exponential, or logarithmic equations or inequalities symbolically or graphically.

* Objectives that describe mathematics content beyond that typically taught in a standard 3-year course of study (the equivalent of 1 year of geometry and 2 years of algebra and statistics).
# Objectives that provide opportunities for questions in the realm of mathematical literacy.

Exhibit 2.6 (continued). Algebra

4) Equations and inequalities (continued)		
Grade 4	**Grade 8**	**Grade 12**
	e) Interpret relationships between symbolic linear expressions and graphs of lines by identifying and computing slope and intercepts (e.g., know that in $y = ax + b$, that a is the rate of change and b is the vertical intercept of the graph).	e) Solve (symbolically or graphically) a system of equations or inequalities and recognize the relationship between the analytical solution and graphical solution.
	f) Use and evaluate common formulas (e.g., relationship between a circle's circumference and diameter [$C = \pi d$], distance and time under constant speed).	# f) Solve problems involving special formulas such as: $A = P(I + r)^t$ or $A = Pe^{rt}$.
		# g) Solve an equation or formula involving several variables for one variable in terms of the others.
		h) Solve quadratic equations with complex roots.

* Objectives that describe mathematics content beyond that typically taught in a standard 3-year course of study (the equivalent of 1 year of geometry and 2 years of algebra and statistics).
# Objectives that provide opportunities for questions in the realm of mathematical literacy.

Chapter 3

MATHEMATICAL PRACTICES

This NAEP framework includes mathematical practices as a fundamental component of the assessment. The inclusion of mathematical practices should not be seen as separate from the mathematics content already assessed by NAEP, but rather as a way to enhance and provide greater insight into what students know and can do in mathematics. Mathematical practices (what one can do with mathematics) are not directly tied to particular instructional practices (how one is taught mathematics). The assessment can be crafted to reach both content and practices and some items may assess content and practice together. Such convergence is often observed in the doing of mathematics, and it is reasonable to expect it in the NAEP assessment of mathematics.

Interest in students' mathematical practices has been growing for over 40 years in mathematics education. Examination of the role of mathematical practices began in the 1980s with a decade of focus on problem solving (NCTM, 1980) and continued throughout the 1990s with an increased attention to student thinking and mathematical discourse in classrooms (e.g., in the standards for mathematics in every state in the U.S. and in NCTM, 2000; 2014).

Over the last two decades mathematics education research has experienced a "social turn" (Lerman, 2000), marked by a shift toward investigating the learning of mathematics in relation to social activity (Adler, 1999; Bell & Pape, 2012; Black, 2004; Civil & Planas, 2004; Ernest, 1998; Enyedy, 2003; Moschkovich, 2007, 2008; NCTM, 1991; van Oers, 2001). A mathematical practice represents what the community writ large values in the patterns of activity that one engages in when doing mathematics. Practices are not at the margins of mathematics. They are – along with content – at the core of mathematics. NAEP is well-positioned to send clear signals concerning the centrality of mathematical practices in what students have an opportunity to learn.

It is now generally agreed that knowing and doing mathematics entails engaging in the practices of the discipline, including generalizing, conjecturing, justifying, mathematizing social contexts, solving problems, communicating, and sense-making (Barbosa, 2006; Goos, 2004; Goos, Galbraith & Renshaw, 2002; Hufferd-Ackles et al., 2004; Hussain et al., 2013; Lau et al., 2009; Truxaw & DeFranco, 2008). As students grapple with and discuss mathematical ideas and problems, they engage in mathematical practices and socialize into mathematics as a discipline (Herbel-Eisenmann & Cirillo, 2009).

This chapter offers a brief overview of mathematical processes and practices as a whole and then describes five key mathematical practices for the NAEP Mathematics Assessment:

- representing;
- abstracting and generalizing;
- justifying and proving;
- mathematical modeling; and
- collaborative mathematics.

As was the case with the content areas in Chapter 2, these five areas are not meant to be exhaustive of all possible mathematical activity. The five NAEP mathematical practices are a particular distillation – for the purposes of assessment – of more than 40 years of development. They reflect a review of current research, national and international assessment frameworks, national standards, and state standards more broadly. These resources, as well as previous NAEP mathematics frameworks, make it clear that assessment of procedural fluency remains important. Also important are the actions involved in selecting, combining, applying, analyzing, and communicating with mathematics in complex problem situations.

Mathematical practices have long been considered important. "Problem solving" has been viewed as a quintessential reason for learning mathematics (NCTM, 1980) as well as being at the core in state standards and subject of the first of the standards for mathematical practice in the CCSS-M (2010). Attention to discourse was a feature of NCTM's *Professional Standards for Teaching Mathematics* (1991) while the use of representations in doing, teaching, and learning mathematics was a process standard in both NCTM's *Curriculum and Evaluation Standards for Mathematics* (1989) and the *Principles and Standards for School Mathematics* (2000). There is considerable agreement on the important mathematical practices across these sources. Note that both the NAEP mathematical practices and the NCTM mathematical process standards explicitly emphasize communication and collaboration, while communication is subtext in several of the CCSS-M mathematical practices (e.g., the "others" in critiquing the reasoning of others, the implicit audiences for a viable argument or expressions about regularity and repeated reasoning, and the tacit stakeholders in a mathematical model).

What may seem odd is that the NAEP mathematical practices do not include problem solving. It would be incorrect to take this as a sign that problem solving is not viewed as an important aspect of mathematics to be assessed by NAEP. Rather this reflects a view of mathematical problem solving as being the synthetic unifying activity of mathematics, encompassing both content and practices. That is, students engage with the NAEP mathematics content and mathematical practices as they solve problems.

A description of each NAEP mathematical practice follows. Although each practice is treated as distinct, they are highly interrelated with one another and with content – as is demonstrated in the examples provided. In designing NAEP items, it may be impossible to completely isolate each mathematical practice. In fact, it may be counterproductive to do so. It would be better to have items that assess content and practices together in natural interplay than to artificially separate the practices for measurement purposes. When items assess multiple aspects of mathematics, it should be possible to identify a primary content focus and a primary practice focus. The former has been done on NAEP Mathematics Assessments for many years, and the latter should be possible moving forward with the mathematical practices.

Just as some mathematics content topics are more likely to interact with others in items, some mathematical practices are more likely to be found in connection with certain mathematics topics. Exhibit 3.20 (at the end of this chapter) suggests where and how the five practices might be assessed within the NAEP mathematics content areas.

Students' mathematical practices are in the purview of NAEP, as a critical component of the mathematics students know and can do. The practices involve both habits of mind and habits of interaction. Habits of mind include such things as building or using mathematical representations, attending to mathematical structure, persevering in solving problems, focusing on reasoning and sense making. Habits of interaction include such things as explaining one's thinking; justifying why a solution works, generalizing a mathematical property, pattern, or process; and raising worthwhile mathematical questions for discussion.

Practice 1: Representing

The term *representation* refers to both process and to product – in other words, to the act of capturing a mathematical concept or relationship in some form and to the form itself. Moreover, the term applies to processes and products that are observable externally as well as to those that occur "internally," in the minds of people doing mathematics (NCTM, 2000, p. 67).

Different representations possess different qualities, which is essentially the reason they exist. For instance, consider the graph of $y = \sin x$. That this can represent a sound wave is not at all obvious from the definition of the sine function in terms of the coordinates of points on a unit circle or as a ratio of lengths of sides of a right triangle. On the other hand, given an oscilloscope readout, the algebra of the sine function becomes a powerful representation. Algebra can be represented using geometry and geometry can be represented using algebra. It is the change in mode from algebraic to geometric (or to numerical or to verbal or to tabular) representation that is linked to viewing one mode as a representation of the other.

Representations are tools for problem solving and representing is used to determine and justify solutions. Students engage in this practice when they create representations themselves, in collaboration with other students, or when they reason from standard representations (e.g., graphs, tables, geometric drawings) that have previously been created by others. Students engage in this practice when they create and/or use visual, contextual, numerical, symbolic, or graphical representations (Lesh, Post, & Behr, 1987).

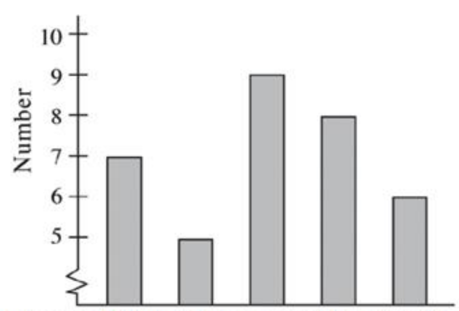
Tripathi (2008) argues that variety in representations "is like examining a concept through a variety of lenses, with each lens providing a different perspective that makes the picture (concept) richer and deeper" (p. 439). According to the National Research Council (NRC, 2009), students, especially young learners, benefit from using physical objects or acting out processes during problem solving. Base 10 blocks (or blocks/tiles representing other bases), fraction strips/bars, red–black integer tiles, and algebra tiles are all examples of physical representations of number and operation that are used to enhance students' understanding of concepts in elementary and middle grades. These visual and physical representations connect, eventually, to symbolic representations as well.

The process of symbolizing mathematical concepts and procedures involves other problem-solving processes as well, such as making connections across various representations, comparing and contrasting multiple approaches to a problem, identifying and using patterns, or generalizing and using mathematical structure. For instance, mathematical representations will inevitably arise in mathematical modelling. Building on the first steps in modelling (to make sense of a situation and identify assumptions), the next step is the attempt to build contextually appropriate representations of the situation. Another example is found in generalizations such as the Pythagorean theorem, which involves mathematical structure and the relationships of the sides of a right triangle, and is associated with one of the most famous symbolic representations in the history of mathematics, $a^2 + b^2 = c^2$, where a and b represent the lengths of the legs of a right triangle, and c is the length of the hypotenuse.

Visual representations play a particularly powerful role in helping students to make sense of problems and understand mathematical concepts and procedures. For instance, beginning at the elementary level, arrays of squares in a grid can be used to represent area models for mathematical operations such as multiplication and division.

Tabular and graphical representations also are important tools for organizing and analyzing quantitative information and relationships among quantities. Students can be asked to examine graphical representations and use them to draw conclusions or make inferences. Exhibit 3.1 is a grade 4 task from the 2005 NAEP Mathematics Assessment. The task involves reading, analyzing, and interpreting graphs within the data analysis and statistics content area. As written, it provides a fixed representation of data and asks students to reason about the given representation. A more nuanced assessment of representing might capture students' thinking based on their own modifications of a given visual representation or examine student response to alternatives to the representation suggested by others. Student creation or modification of representations can promote discourse and opportunities for students to critique and debate various approaches to problems. Representations therefore can provide additional objects and incentives for students to demonstrate collaboration and communication of their thinking about the mathematics in an assessment.

Exhibit 3.1. Grade 4 NAEP Data Item

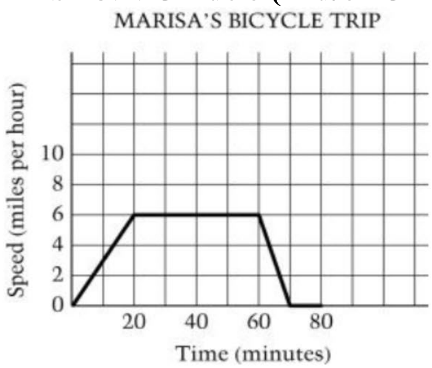


Jim made the graph above. Which of these could be the title for the graph?

A. Number of students who walked to school on Monday through Friday
B. Number of dogs in five states
C. Number of bottles collected by three students
D. Number of students in 10 clubs

In Exhibit 3.1 a representation is provided for student consideration. Exhibit 3.2 suggests how students might modify a given representation, or generate several alternative representations based on a scenario. The bicycle trip problem from the 2003 NAEP Mathematics Assessment shown in Exhibit 3.2 is a task where the student is asked to take a given representation and work backwards to a context that could fit that representation.

Exhibit 3.2. Grade 8 (and/or Grade 12) NAEP *Bicycle Trip* Item



The graph above represents Marisa's riding speed throughout her 80-minute bicycle trip. Use the information in the graph to describe what could have happened on the trip, including her speed throughout the trip.

During the first 20 minutes, Marisa ____

____

____

From 20 minutes to 60 minutes, she ____

From 60 minutes to 80 minutes, she ____

____

____

The item also could be rewritten to include a different graph of the bicycle trip where the speed is zero at some point mid-ride and constant for a shorter period of time. Also, the graph might have a more realistic range of speeds, up to 12 mph. Subsequently, students could be given a correct explanation provided by another (hypothetical) student, and asked to provide another, alternative explanation for how the bike rider's trip could have generated a graph like this. Alternatively, they could be given two different explanations by hypothetical students, and asked to decide if either or both explanations correctly match the representation in the graph.

In this approach, students would be given the graph of the speed over time and asked to think about the story of what was happening on the ride. However, the reverse situation could also be used to build a scenario-based task for Marisa's ride. Students could be given a story (in writing or possibly a short animation or video) about Marisa's ride, and then be asked to create their own graphical representation of the ride over time, connecting key features of the story with features of the representation in the graph.

Practice 2: Abstracting and Generalizing

An essential element of mathematical problem solving is the ability to reason abstractly and to develop, test, and refine generalizations. In reasoning abstractly, students engage in the process of *decontextualizing*: Abstracting ideas in a given problem or context and expressing and manipulating them in a manner independent of their contextual references. Decontextualizing can foster an understanding of the relationships among problem contexts and written or symbolic forms, as well as an understanding of how mathematical expressions might be transformed to facilitate a solution strategy.

Reasoning abstractly also includes the processes of reasoning with mathematical structures, that is, with mathematical elements that can be combined in particular ways to form a coherent whole. Harel and colleagues (Harel, 2018; Harel & Soto, 2016) have defined *structural reasoning* as the ability to look for, recognize, probe into, and make general mathematical structures. It relies on an understanding of a structure as being something made up of a number of parts that are put together in a particular way. These parts can include elements of an algebraic expression, words or phrases in a problem situation, or operations across different mathematical domains, such as the multiplication of whole numbers and the multiplication of binomial expressions. Examples of structural reasoning include reducing unfamiliar structures into familiar ones, carrying out operations mentally without performing them, and reasoning with abstracted or generalized representations instead of specific instances or examples.

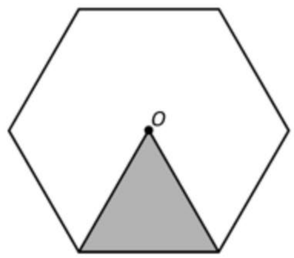
For instance, young students can notice patterns of additive commutativity, such as three plus seven yielding the same sum as seven plus three. In this instance, decontextualization would include finding a way to represent this relation independent of particular numbers, as a more general identity: $a + b = b + a$. Another example of structural reasoning is recognizing similar mathematical structures across different problems or domains. For example, one could see the multiplication of two binomials $(2x + 7)(3x + 2)$ as a more general instantiation of multiplying 27 by 32.

Consider the 2017 NAEP grade 8 Geometry item in Exhibit 3.3. This item requires students to express the area of the hexagon in terms of the area of the given shaded triangle. Students are then asked to extend their reasoning to a 10-sided figure. Thus, students are first challenged to reason structurally by mentally comparing the area of the triangle formed by the hexagon’s center and two adjacent vertices with the area of the entire figure. Then, students are further tasked with extending their reasoning from the specific case of the hexagon to another regular polygon.

Exhibit 3.3. Grade 8 NAEP Geometry Item

Point *O* is the center of the regular hexagon shown.

The shaded triangle is formed by *O* and two adjacent vertices of the hexagon and has an area of *T*.



What is the area of the hexagon in terms of *T* ?

Area=

Point *P* is the center of a regular polygon with 10 sides.

A triangle is formed by *P* and two adjacent vertices of the polygon and has an area of *V*.

What is the area of the polygon in terms of *V* ?

Area=

Although a student could solve the problem in Exhibit 3.3 by drawing a 10-sided polygon, the specified triangle, and then counting the number of triangles that comprise the polygon, a student could also carry out this operation mentally rather than drawing it out. Also, the item could be revised to elicit decontextualizing, thinking about the relationship between the specified triangle and any regular polygon. In the later grades, students could be expected to engage in the practice of abstracting and generalizing, expressing their reasoning algebraically and extending it to justify and prove a conjecture about the general relationship between the triangle and any *n*-sided regular polygon.

As demonstrated in the above examples, abstracting can occur across different domains. Abstracting can be assessed in reasoning about figures and their relationships in geometry, about number theory in number properties and operations, or about equivalence or functional relationships in algebra. The manner in which one decontextualizes or reasons with structure will differ across the domains, but these are processes students can employ in any of the content areas included in the NAEP Mathematics Assessment.

Mathematics education researchers and policymakers have defined generalizing in a number of ways. Historically, generalization has been defined as an individual, cognitive construct (e.g., Carraher, Martinez, & Schliemann, 2008). These definitions characterize generalization as the act of identifying a property that holds for a larger set of mathematical objects or conditions than the number of individually verified cases. For instance, Harel and Tall (1991) described generalization as the process of "applying a given argument in a broader context" (p. 38), and Radford (2006) argued that generalization involves identifying a commonality based on particulars and then extending it to all terms (i.e., structural reasoning).

More recently, researchers have begun to address generalizing as a construct that is both social and cognitive, rooted in activity and context. From this perspective, generalization is informed by social interaction, history, and artifacts and can occur individually or collectively (Jurow, 2004). Generalizing is viewed as a practice rooted in, and mediated by, discourse and activity.

Drawing on this perspective, generalizing is an individual or collective practice of (a) identifying commonality across cases, (b) extending reasoning beyond the range in which it originated, and/or (c) deriving broader results from particular cases (Ellis, 2007).

There are a number of aspects of mathematical reasoning that can foster generalizing. These include visualizing, focusing, reflecting, connecting, and expressing. *Visualizing* involves perceiving patterns or structural relationships, as well as imagining a set of relationships beyond what is perceptually available. *Focusing* is attending to particular details, characteristics, properties, or relationships above others. This can include specializing on a particular case in a pattern or attending to figural or numerical cues. *Reflecting* involves thinking back on the operations one has carried out, observing one's method in solving problems, or examining the rules that govern a given pattern. *Connecting* is the identification of relationships among tasks, representations, or properties. Making connections between representations or identifying and operating on structural similarities can foster the development of generalizations. Finally, *expressing* is depicting a generalization verbally or in written language. Describing generalizations in words can support the subsequent development of algebraically-represented generalizations.

Like abstracting, generalizing can occur across the content domains and grade bands. Existing NAEP Mathematics Assessment items contain a number of generalization tasks, in which students are asked to determine the rule guiding the pattern of number terms in a sequence. The potential rules are provided for students, and for this item type students are prompted only to attend to the action required to move from one term in the sequence to the next. In other items, students must determine the rule themselves, such as for the 2013 NAEP grade 4 item in Exhibit 3.4 and 2005 NAEP grade 12 item in Exhibit 3.5

Exhibit 3.4. Grade 4 NAEP Number Pattern Item

3, 4, 6, 9, 13, ...

The growing number pattern above follows a rule.

Explain the rule.

Write a new growing pattern beginning with 21 that follows the same rule.

21, _______ , _______ , _______ , _______ , _______

Exhibit 3.5. Grade 12 NAEP Number Pattern Item

Sequence I: 3, 5, 9, 17, 33, . . .

Sequence I, shown above, is an increasing sequence. Each term in the sequence is greater than the previous term.

a. Make a list of numbers that consists of the positive differences between each pair of adjacent terms in Sequence I. Label the list Sequence II.

b. If this same pattern of differences continues for the terms in Sequence I, what are the next two terms after 33 in Sequence I ?

6th term ______________________

7th term ______________________

c. Write an algebraic expression (rule) that can be used to determine the n^{th} term of Sequence II, which is the difference between the $(n+1)^{\text{st}}$ term and the n^{th} term of Sequence I.

Notice that for the grade 12 item, students are expected to write a formal algebraic rule for moving from the n^{th} term to the $(n + 1)^{\text{st}}$ term. In other items, students are tasked with determining an explicit rather than a recursive rule to find the n^{th} term in a sequence.

The above items present fairly typical generalization tasks, in which students are presented with a sequence of numbers or figures and must generalize either a recursive or explicit relationship. However, students can also be challenged to engage in the processes of generalizing in other items that do not rely on pattern sequences, as in the scenario in Exhibit 3.6 (next page).

The task supports a number of possible generalizing processes, as well as the opportunity for abstracting. For instance, one could consider that for each coin (nickel, dime, quarter), there are two possible outcomes, H or T. Thus, a student could either systematically list outcomes to determine that there are 8 total outcomes or could begin to think structurally to reason that for three coins and two outcomes per coin, there must be $2^3 = 8$ total outcomes. Alternatively, through systematic listing a student could determine that there are 1 + 3 + 3 + 1 outcomes, corresponding to 1 outcome with exactly zero Ts, 3 outcomes with exactly one T, 3 outcomes with exactly two Ts, and 1 outcome with exactly three Ts. Extending to the 4-coin case, for instance, students might determine that the number of outcomes is 1 + 4 + 6 + 4 + 1, corresponding to 1 outcome with exactly zero Ts, 4 outcomes with exactly 1 T, 6 outcomes with exactly 2 Ts, 4 outcomes with exactly three Ts, and 1 outcome with exactly four Ts (and symmetrically but opposite for the number of Hs).

Exhibit 3.6. Grade 8 and/or Grade 12 task (adapted from 2013 grade 8 NAEP item).

Three students each have a coin, one has a nickel, one has a dime, and the third student has a quarter. They flip their coins at the same time. Each coin can land either heads up (H) or tails up (T). List all the different possible outcomes for how the coins could land in the chart below. The list has been started for you.

Nickel	Dime	Quarter
H	H	H
H	H	T

What if a 4th student joins the group with a half-dollar coin? How many different ways could the 4 coins land? What if a 5th student joined with a penny–how many different ways could the 5 coins land?

One aspect of generalizing is identifying commonality across cases, and students might notice that the outcomes for the 3-coin and 4-coin scenarios can be structured according to the rows in Pascal's triangle. Or, students might reason that, like the 3-coin case, each of the positions in the 4-coin case has two possible outcomes, H or T, and thus the total number of possible outcomes must be $2^4 = 16$, and more generally for n coins, 2^n. Such a task could afford a number of rich generalizing opportunities, regardless of whether students are expected to recognize that 2^n is the sum of the coefficients of the binomial $(a+b)^n$ (e.g., $2^4=1+4+6+4+1$).

Abstracting and generalizing support students' problem-solving activity. The types of structural elements students identify and abstract will influence the generalizations they make, and students' processes of generalization can, in turn, affect other aspects of problem solving.

Practice 3: Justifying and Proving

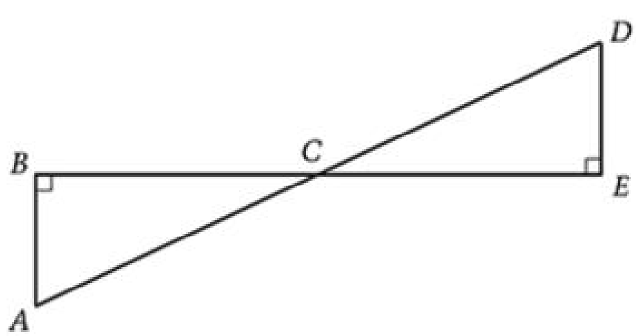
Justifying and proving are key aspects of mathematical activity in all topics and grade levels. Traditionally, proof was viewed as a form of mathematical argumentation pertaining first to high-school geometry and not visited again until pre-calculus courses with proofs of trigonometric identities and proofs by mathematical induction. However, this changed in the latter half of the 20th century. The *Principles and Standards for School Mathematics* emphasized the importance of justifying and proving at all levels of mathematics, noting that "reasoning and proof should be a consistent part of students' mathematical experience in prekindergarten through grade 12" (NCTM, 2000, p. 56). Similarly, state standards and the CCSS-M (2010) highlight the activities students engage in as they learn to create valid mathematical arguments: making and investigating conjectures, developing particular forms of argument (e.g., deductive, inductive), and using a variety of proof methods (e.g., direct, counterexample). These are all considered components of the practice of justifying and proving.

Mathematical justification includes creating arguments, explaining why conjectures must be true or demonstrating that they are false, exploring special cases or searching for counterexamples, understanding the role of definitions and counterexamples, and evaluating arguments. A valid justification should show why a statement or conjecture is true or not true generally (e.g., for all cases) and, at the later grades, does so by providing a logical sequence of statements, each building on already "known to be true" statements, ideas, or relationships.

A justification is *not* based on authority, perception, popular consensus, or examples. As students engage in justifying, they may be tempted to rely on external sources to verify their ideas, such as their teacher or a textbook (Harel & Sowder, 1998). They may also want to use examples to support their claims, concluding that a conjecture must be true because it holds for several different cases. Examples can and do play an important role in justifying and proving, particularly in terms of helping students make sense of statements and gain a sense of conviction, but they do not suffice as a mathematical justification or proof except for proofs by exhaustion or proofs by counterexample.

A formal proof is a specific type of argument "consisting of logically rigorous deductions of conclusions from hypotheses" (NCTM, 2000, p. 55). In grade 12, students are expected to develop formal mathematical proofs. A proof uses definitions and theorems that are true and available without further justification; a proof is therefore valid only if the assumptions upon which it relies have already been shown to be true.

Often, the phrase "mathematical proof" conjures an image of the traditional two-column proof that is typical in high-school geometry classrooms. This form of proof can be helpful for supporting students' efforts to develop a clear chain of statements, each relying on the prior, and for making sure that each statement is justified. However, proofs can take on many different forms. For instance, consider the 2009 NAEP grade 12 geometry item in Exhibit 3.7.



Given: C is the midpoint of $\overline{BE}$.
$\angle B$ and $\angle E$ are right angles.

Exhibit Prove that $\overline{AC} \cong \overline{DC}$ and give a reason for each statement in your proof.

This item lends itself well to a two-column proof, particularly because it stipulates that a reason must be provided for each statement in the proof. One proof that $\overline{AC} \cong \overline{DC}$ is as follows:

Statement	Reason
C is the midpoint of $\overline{BE}$	Given
$\angle B$ and $\angle E$ are right angles	Given
$\overline{BC} \cong \overline{EC}$	Definition of midpoint
$\angle B \cong \angle E$	Right angles are congruent
$\angle ACB \cong \angle DCE$	Vertical angles are congruent
$\triangle ACB \cong \triangle DCE$	Angle-Side-Angle (or Leg-Angle)
$\overline{AC} \cong \overline{DC}$	Corresponding parts of congruent triangles are congruent

However, there is nothing about the prompt that stipulates that the proof must occur in a two-column format. A proof can have many different forms, including narrative form, picture, diagram, two-column, or algebraic forms. The form used to represent a mathematical proof is valid as long as it communicates the essential features of the proof, namely, that it contains logically connected mathematical statements that are based on valid definitions and theorems.

A narrative form of the proof in answer to the item in Exhibit 3.7 could also be appropriate, as seen below:

> The measures of $\angle BCA$ and $\angle ECD$ are equal because vertical angles have the same measure. We also know that the measures of $\angle B$ and $\angle E$ are the same because they are both right angles. Since C is the midpoint of $\overline{BE}$, $\overline{BC} \cong \overline{EC}$. So, by the angle-side-angle theorem, triangle ACB is congruent to triangle DCE. Therefore, $\overline{AC} \cong \overline{DC}$ because corresponding parts of congruent triangles are congruent.

In addition to the various formats one can use to develop or present proofs, there are also many ways of mathematically proving or disproving. These include, among others, developing deductive arguments, finding counterexamples, engaging in proof by exhaustion, and employing mathematical induction. Often, it may be easier to use a particular mode of argumentation based on the nature of the claim. Understanding counterexamples is a particularly important element of justifying and proving. The process of refuting - demonstrating that a particular statement is false - is a key element of justification because conjecturing can produce both true and false statements. Students must understand that a single counterexample disproves a generalization.

An example of the value of finding a counterexample can be seen in the grade 12 algebra item in Exhibit 3.8. Here, one could find a value for x that is, for instance, less than 5 but not also greater than -3 (e.g., $x = -10$). That single counterexample is sufficient to show that Dave's claim cannot be correct, for $x = -10$ does not satisfy the statement $-3 < x < 5$.

Exhibit 3.8. Grade 12 NAEP Algebra Counterexample Item

> Question A: If x is a real number, what are all values of x for which $x > -3$ <u>and</u> $x < 5$?
>
> Question B: If x is a real number, what are all values of x for which $x > -3$ <u>or</u> $x < 5$?
>
> Barbara said that the answers to the two questions above are different.
>
> Dave said that the answers to the two questions above are the same.
>
> Which student is correct?
>
> ◯ Barbara ◯ Dave
>
> Explain why this student is correct. You may use words, symbols, or graphs in your explanation.

Similarly, only one counterexample is needed to refute Pat's claim in the grade 8 number properties and operations item in Exhibit 3.9. Multiplying 6 by any real number less than 1 will yield a result less than 6, confirming Tracy's claim and refuting Pat's claim.

Exhibit 3.9. Grade 8 NAEP Number Properties and Operations Counterexample Item

> Tracy said, "I can multiply 6 by another number and get an answer that is smaller than 6."
>
> Pat said, "No, you can't. Multiplying 6 by another number always makes the answer 6 or larger."
>
> Who is correct? Give a reason for your answer.
>
> ____
>
> ____
>
> ____

Understanding that a single counterexample undermines a general claim is an important but difficult aspect of justification. Learning to search for counterexamples and explaining why they are justifications is only one aspect of refutation. The process of attempting to prove that a conjecture is false can also lead to the development of new insights or ideas, as well as forming different conjectures that can then be explored and proved.

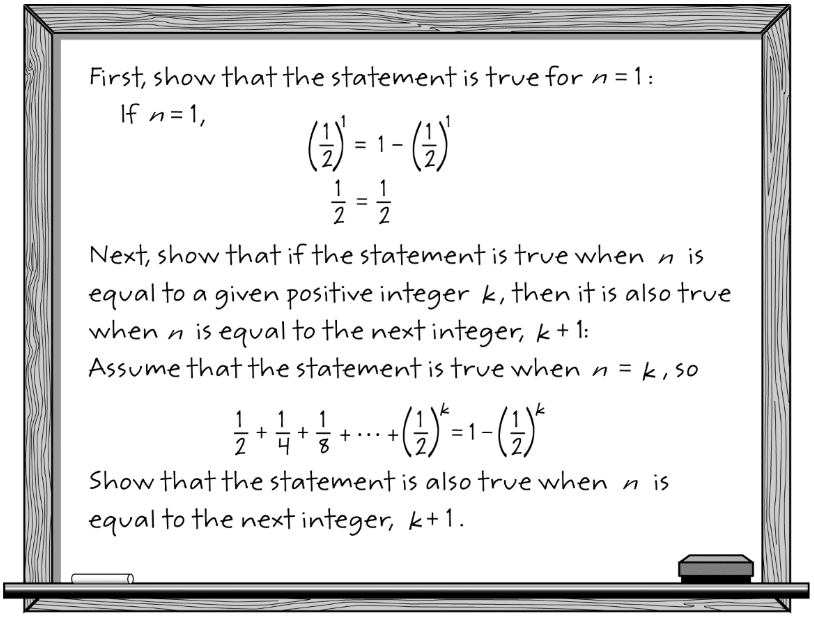
Some NAEP items require a particular mode of proof, such as the grade 12 number properties and operations item in Exhibit 3.10.

Exhibit 3.10. Grade 12 NAEP Number Properties Mathematical Induction Item

A student was asked to use mathematical induction to prove the following statement.

$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots+(\frac{1}{2})^n = 1-(\frac{1}{2})^n$ for all positive integers n

The beginning of the student's proof is shown below.



Complete the student's proof by showing that if the statement is true when $n = k$, then it is also true when $n = k + 1$, where k is any positive integer.

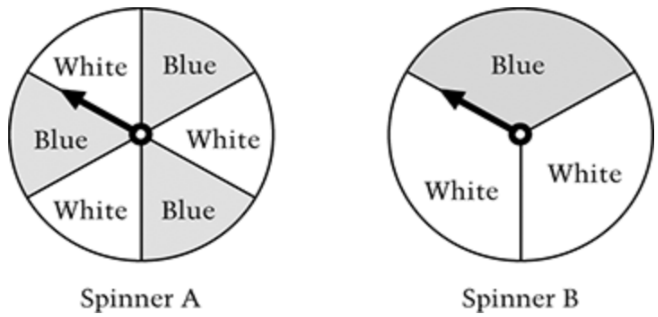
Here, a student must use the tools of mathematical induction to complete the provided argument:

For $n = k + 1$, $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots+\left(\frac{1}{2}\right)^{k+1}$ can be expressed as $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots+\left(\frac{1}{2}\right)^{k}+\left(\frac{1}{2}\right)^{k+1}$. We know from the above statement that $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots+\left(\frac{1}{2}\right)^{k}$ is equal to $1-\left(\frac{1}{2}\right)^{k}$, so substituting that yields $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots+\left(\frac{1}{2}\right)^{k+1} = 1-\left(\frac{1}{2}\right)^{k}+\left(\frac{1}{2}\right)^{k+1}$. Simplifying the expression on the right gives us $\frac{2^{k+1}-1}{2^{k+1}}$, or $1-\left(\frac{1}{2}\right)^{k+1}$.

Being familiar with a variety of approaches to generating a proof, and knowing which one to select for a particular circumstance, is an important aspect of justifying and proving.

Another element of the mathematical activity in justifying and proving is evaluating the validity of a purported proof. This involves not only deciding whether a proof is valid in terms of its conclusion. It includes determining whether a given proof relies on correct assumptions, makes use of merited conclusions and logic, and explains the entire statement or conclusion. These skills can be fostered by challenging students to judge the appropriateness of a given argument (e.g., a formal or informal proof; Knuth, Choppin, & Bieda, 2009). Some NAEP items could be adjusted or expanded to include evaluation of the justifying and proving of others. For instance, the grade 4 data analysis, statistics, and probability item in Exhibit 3.11 addresses the question of maximizing the probability of landing on blue.

Exhibit 3.11. Grade 4 NAEP Probability Spinners Item



Lori has a choice of two spinners. She wants the one that gives her a greater probability of landing on blue.

Which spinner should she choose?

○ Spinner A ○ Spinner B

Explain why the spinner you chose gives Lori the greater probability of landing on blue.

Asking students to explain *why* the spinner they chose gives Lori the greater probability of landing on blue would foreground justifying. Students could also be given a scenario version of this task in which other students' explanations for choosing spinner A are provided, and then be asked which of the explanations is the most convincing to them and *why* it convinces them. Versions of the examples below might be offered as text, or by avatars, or through video.

1. Andreas says Spinner A has a greater chance for landing on blue because it has three blue sections and Spinner B only has one blue section.
2. Basil says that Spinner A will have a greater probability of landing on blue because the area of two of the blue sections on Spinner A is equal to the area of the one blue section on Spinner B.
3. Calista says that Spinner A has a greater chance of landing on blue because she tried it out. Calista spun each spinner 10 times. For Spinner A, the arrow fell on blue 6 times. For Spinner B, it only fell on blue 2 times.
4. Dora says that Spinner A will have a greater probability because it is one-half blue but Spinner B is only one-third blue and one-half is more than one-third.

Engaging in justifying and proving is a way for students to explore why a particular assertion must be true. Granted, some proofs might only serve to verify the truth of a statement without helping students understand why; researchers refer these as "proofs that prove" rather than "proofs that explain" (Hanna, 1983). Certainly not all proofs are explanatory, but in many cases, justifying or evaluating a given argument can help students gain insight into why a conjecture is true. Investigating why a conjecture holds can support students' to attend to particular features that may provide insight into relationships, examine multiple factors that are relevant to the problem statement, return to the meanings of terms and operations, or consider similarity or difference across cases. By exploring these factors, students gain new insight into the conjecture or deepen their understanding of fundamental mathematical ideas.

The grade 8 algebra item in Exhibit 3.12 foregrounds generalizing. The pattern, that the number of diagonals d is equal to the number of sides $n – 3$, is readily apparent from the provided cases. However, adding a prompt for justifying *why* $d = n - 3$ is a reasonable conjecture or a prompt about proving why the statement is true for any convex polygon would foreground justifying and proving. To create the justification that would answer why $d = n - 3$ might involve drawing a few cases, reasoning that from any given vertex one cannot draw a diagonal to itself and one cannot draw a diagonal to the two adjacent vertices (because this makes up two of the sides of the polygon) and noting that the three vertices cannot have diagonals drawn to them while the remaining vertices can.

Exhibit 3.12 Grade 8 NAEP Algebra Generalization Item

From any vertex of a 4-sided polygon, 1 diagonal can be drawn.

From any vertex of a 5-sided polygon, 2 diagonals can be drawn.

From any vertex of a 6-sided polygon, 3 diagonals can be drawn.

From any vertex of a 7-sided polygon, 4 diagonals can be drawn.

How many diagonals can be drawn from any vertex of a 20-sided polygon?

Answer:_________________

The item in Exhibit 3.12 also could be revised into a task to justify why the total number of diagonals that can be drawn for any given convex polygon is $n(n - 3) / 2$. Justifying could take the form of describing why the number of diagonals that can be drawn from a vertex is $n - 3$ (as above) and noting that one could draw $n(n - 3)$ diagonals. However, this would mean that each diagonal would be drawn twice, to and from each vertex. Therefore, in order to avoid double counting the diagonals, one must divide by 2, yielding the expression $n(n - 3) / 2$. To further illustrate the difference between a proof that proves and a proof that also explains, note that this expression for the total number of diagonals can also be proved by induction. Such a proof by induction would verify the statement without revealing why it is true.

Justifying and proving can help students develop a new and deeper understanding of the mathematics content at hand. Making sense of others' justifications or proofs and determining their validity can help students generate new ideas, conjectures, and generalizations, or can support their efforts to develop a new theory to be tested. That is, justifying and proving is an important mode of communication. Proofs can reveal the tools, strategies, modes of thinking, and resources used by those who created them.

Practice 4: Mathematical Modeling

Mathematical modeling has been defined as "a process that uses mathematics to represent, analyze, make predictions or otherwise provide insight into real-world phenomena" (Society for Industrial and Applied Mathematics & Consortium for Mathematics and Its Applications [SIAM & COMAP], 2016, p. 8). The importance of the practice of mathematical modeling is reflected in its inclusion in most state standards (Johnston et al., 2018) and as one of the eight standards for mathematical practice in the CCSS-M (2010). It is also the focus of the *Guidelines for Assessment and Instruction in Mathematical Modeling Education*, which includes attention to the "team sport" nature of much mathematical modeling activity:

> Mathematics is sometimes seen as a solitary activity, perhaps reinforced by our evaluation of individual efforts in school and in competitions. Modeling is an inherently team sport and the problems are big and messy enough that a team approach helps students find useful solutions. The job skills of work distribution, communication (including listening), and cooperation can all come into play naturally as a group works together toward their solution. (SIAM & COMAP, 2016, p. 27)

At an introductory level, modeling involves steps such as selecting and applying particular mathematical processes to solve a problem or representing mathematical concepts and processes (such as mathematical operations) using visual, physical, or symbolic representations. At a more advanced level, a series of processes may be needed to mathematize a messy real-world situation prior to selecting and applying the mathematics, and then follow-up work involves analyzing and evaluating the results obtained from doing the mathematics. A full cycle in the mathematical modeling process includes (a) identifying the problem; (b) making assumptions and identifying variables; (c) mathematizing the situation; (d) analyzing and assessing solutions; (e) iterating the process; (f) implementing the model; and (g) reporting out results (SIAM & COMAP, 2016).

Mathematical modeling involves more than having students either add context to a decontextualized mathematics problem or solve an applied mathematics problem. Instead, modeling involves student choice—including the assumptions made and the posing of answerable questions given an open ended situations. Thus, the practice of modeling requires students make sense of a scenario, mathematize it, and apply the mathematization to reach and check the viability of a solution. Moreover, mathematical modeling requires discussions and decisions about what is valuable (Burroughs & Carlson, 2019).

In a NAEP assessment context, it is rare that students would have the time to go through all the steps in the modeling process. However, mathematical modeling can be assessed with increasing attention to detail in sub-parts of the modeling process. For example, given an open-ended situation, students could generate questions they would need to explore or identify some assumptions needed in order to begin the modeling process. In such scenarios, students would engage in the first two steps of the modeling process.

Scenario-based tasks are particularly useful in assessing student achievement in the practice of mathematical modeling. Consider the Lunch Problem task in Exhibit 3.13. It is adapted from a scenario posed in the *Guidelines for Assessment and Instruction in Mathematical Modeling Education* [GAIMME] (SIAM & COMAP, 2016, pp. 32-35).

The school is trying to decide on the numbers of fruits and vegetables to put out for a school lunch. Their choices are carrots, celery, cherry tomatoes, oranges, bananas, and apples. How many of each should they put out to offer the best lunch for the students?

Exhibit 3.13. Grade 4 GAIMME Lunch Problem Scenario

For this scenario-based task, students could be asked to work in pairs or in teams or to interact with a virtual team as they model the given open-ended lunch choice situation. In the process of mathematizing the scenario, some questions students may need to consider are: “How many students are in the school? Do students like some of these choices more than others? Do some of these choices cost more than others? If so, which ones might we have some left over, which might we run out of? Should the school’s cost of these items be considered?” Students who address these questions would be identifying choices (variables) to be included in lunch and making certain assumptions about those variables, including some boundaries within which to try to solve the lunch problem.

A NAEP assessment task could be posed in different ways depending on grade levels. Grade 8 students could be given the final prompt: “How many and what types of pizzas should be ordered for the 8th grade party?” Some possible questions for students to address as they attempt to model this situation are: “How many students do we expect to feed? How can we find out what types of pizza they like? Should we survey some of the students? How do we decide who to survey? What size pizzas should we order? What is the cost of each size of pizza?” Here students have to devise survey questions (identify the problem), narrow down to choices of pizza and sizes of pizza (make assumptions), and as soon as they begin to investigate costs of sizes and types of pizza they can begin to build model estimates for the cost of the party.

At grade 12, an item or set of items might be developed around a scenario such as: “What is the best type of computer for the school district to order for student to use in computer labs?” Some possible issues students may need to address as they attempt to model this situation are: “How many computers are needed in a school lab, and how do we know? Is there a break on cost if a large number of computers are purchased at the same time? Which types of classes will need access to the computers? What types of software will be needed for the classes? Do any of the companies offer deals for software along with the computer purchase? How much money can we spend per student?” There are many up-front decisions to be made about what to include and what to assume to address this task. The problem also evokes initial mathematization processes when students ask questions like: “How much money per student?” or “Are there deals for software inclusion or a price break on a large order?”

As another example, the existing NAEP income tax item in Exhibit 3.14 could be posed in a scenario-based form as a modeling task, see Exhibit 3.15.

Exhibit 3.14. Grade 12 NAEP Algebra Income Tax Item

This question requires you to show your work and explain your reasoning. You may use drawings, words, and numbers in your explanation. Your answer should be clear enough so another person could read it and understand your thinking. It is important that you show all your work.

One plan for a state income tax requires those persons with income of \$10,000 or less to pay no tax and those persons with income greater than \$10,000 to pay a tax of 6 percent only on the part of their income that exceeds \$10,000.

A person's effective tax rate is defined as the percent of total income that is paid in tax.

Based on this definition, could any person's effective tax rate by 5 percent? Could it be 6 percent? Explain your answer. Include examples if necessary to justify your conclusions.

Exhibit 3.15. Example of a Scenario-based Task - Income Tax Modeling

In one state the tax plan is for residents to pay a 6% tax on all income over \$10,000, no tax on any income of \$10,000 or less. The *effective tax rate* is defined as the percent of your *total income* that is paid in tax.

Create a formula that can be used to calculate the *effective* tax rate of your total income in this state. What is the highest *effective tax rate* that a person could pay? Use your model to defend your position on the highest possible rate.

Exhibit 3.15 is an example where some initial information is provided, and students could work in teams to develop a mathematical model. The task as posed primarily calls on mathematizing, as well as analyzing and assessing. It involves building a general symbolic model for the *effective tax rate* (ETR). ETR can be expressed as the ratio of tax T to income I, or T/I. Students first need to compute the tax on income I, with the given 6% rate over the first \$10,000 of income, arriving at $T = .06(I - \$10{,}000)$. Then creating a symbolic model for the effective tax as ETR = $T/I = .06(I - 10{,}000)/I$.

To answer questions about the highest possible tax rate, students could create a graphical model of ETR as a function of I. The mathematization process for this task starts with decisions about using ratios and percent (grade 8 tasks), and then evolves to developing an algebraic expression to model ETR, and eventually to a graph of ETR as a function of I, and the analysis of the graph to support the argument that there is an upper bound for the value of ETR in the model (grade 12 tasks).

Another illustrative problem involves data modeling. The task in Exhibit 3.16 is an example from the on-line bank of tasks available from the Levels of Conceptual Understanding in Statistics (LOCUS) project.

> The student council members at a large middle school have been asked to recommend an activity to be added to physical education classes next year. They decide to survey 100 students and ask them to choose their favorite among the following activities: kickball, tennis, yoga, or dance.
>
> What question should be asked on the survey? Write the question as it would appear on the survey.
>
> Describe the process you would use to select a sample of 100 students to answer your question.
>
> Create a table or graph summarizing possible responses from the survey. The table or graph should be reasonable for this situation.
>
> What activity should the student council recommend be added to physical education classes next year? Justify your choice based on your answer to part (c).

Exhibit 3.16 Grade 8 LOCUS Modeling a School Activity Problem

As posed, this task closely follows the four-stage statistical investigation process as outlined by Franklin and colleagues (2007): (a) identifying a statistical question for investigation; (b) gathering appropriate data; (c) analyzing the data; and (d) communicating the results. The task assesses a number of the content objectives in the Data Analysis, Statistics, and Probability area, including posing a statistical question, addressing issues of bias in surveys, creating tables and graphical representations of data. Although the task as stated covers the entire modeling cycle (SIAM & COMAP, 2016), parts of the task could be already supplied to students and they could be asked to complete the next step(s) in the modeling process.

Although modeling tasks – especially separate aspects of the modeling process – could be posed to individual students, in the workplace mathematical modeling is often done in teams. The importance of preparing students to solve problems is regularly identified as a 21st century skill. The U.S. Department of Labor has noted:

> The ability to work as part of a team is one of the most important skills in today's job market. Employers are looking for workers who can contribute their own ideas, but also want people who can work with others to create and develop projects and plans. (U.S. Department of Labor, ODEP, p. 57)

Modeling provides an inviting context for the use of collaborative tasks that are addressed by groups of students working together. In school mathematics, students already often work together in groups on mathematical tasks. The practice of mathematical modeling is a natural place to use scenario-based tasks in the NAEP Mathematics Assessment. Many of the sample tasks provided in this section could best be done by groups or pairs students. When a task is worthy of group effort, the assessment could focus on group responses, group solutions, and group problem solving activity. Such an assessment approach is central to the final practice of the NAEP Mathematics Framework, collaborative mathematics.

Practice 5: Collaborative Mathematics

As a social enterprise, doing mathematics with others involves pooling ideas, argumentation, and collaborative problem solving whereby ideas are offered, connected, and built-upon toward solution and shared understanding. Thus, learning and doing mathematics involves both individual and collaborative processes. Drawing on the NAEP Technology and Engineering Literacy (TEL) and PISA frameworks, which both assess collaboration, as well as the research literature on collaborative mathematics, this practice is defined in relation to the establishment of joint thinking among individuals toward the construction of a problem solution.

Collaborative mathematics refers to the talk and actions students engage in with one another as they engage in a necessary collaboration – where the mathematics is too complex or messy for an individual to meet its demands alone (Fiore et al., 2017). In this sense, collaborative mathematics (including mathematical discussions and collaborative problem solving) is a social interaction that draws on and influences mathematics discourse practices (Chapin, O'Connor, & Anderson, 2009; O'Halloran, 1998). Rather than seen as arising from general psychological processes, such as metacognition and problem solving skills, a discourse perspective grounds these collaborative activities as fundamentally linked to the mathematics – both arising from and shaping mathematical knowledge and action.

As a practice, collaborative mathematics exists alongside other mathematical practices. That is, as students work together towards a shared goal, they may also engage in representing and symbolizing, abstracting and generalizing, justifying and proving, and mathematical modeling. Assessing collaborative mathematics requires developing items that foreground and require the doing of mathematics collaboratively in nature, on processes that are fundamentally about *joint thinking*. Collectively, these processes include sharing ideas with others; attending to and making sense of the mathematical contributions of others; evaluating the merit of others' ideas through agreement or disagreement; and productively responding to others' ideas through building on or extending ideas and connecting or generalizing across ideas.

Collaborative mathematics processes are largely understood as discursive in nature and occurring through social interaction during mathematical activity. The NCTM's policy documents have maintained a long-standing focus on discourse and communication. Beginning with the Mathematics as Communication standard (NCTM, 1989) and attention to discourse (NCTM, 1991), mathematics educators have argued that when students write and talk about their thinking, they not only clarify their own ideas, but they also offer valuable information for assessment. In the *Principles and Standards for School Mathematics* (NCTM, 2000), reflection and communication are seen as intertwined processes in mathematics learning, and it is argued that when ideas are worked out in public, not only do students benefit, but teachers can better monitor student thinking and learning (Staples, 2007).

Given the discursive nature of collaborative mathematics, NAEP Mathematics Assessment items that measure collaborative processes should likewise be discursive in nature, offering students examples of social interaction or imagined utterances around mathematics to which they are tasked to respond in key ways. These include being asked to make sense of others' thinking, express and defend agreement or disagreement, and extend an idea. Tasks might also be genuinely collaborative in nature, asking assessed students to work together in a team during the assessment, such as on a mathematical modeling task.

PISA assesses a related idea: collaborative problem solving. For PISA this is defined as “the capacity of an individual to effectively engage in a process whereby two or more agents attempt to solve a problem by sharing the understanding and effort required to come to a solution and pooling their knowledge, skills, and efforts to reach that solution” (OECD, 2017, p. 6).

As illustrated in the components from a PISA scenario-based collaborative problem-solving task (Exhibits 3.17 and 3.18), the structure is as a dialogue with a team constituted by avatars and the assessed student. The problem task is on the right of the screen, while the running dialogue is on the left (Exhibit 3.17). The assessed student is to choose a discursive response to productively move the collaboration forward. In the example offered in the subsequent screenshots in Exhibit 3.18, one can see that the item emerges as interactional contributions are offered by each avatar (“Brad” and “Rachel”) and the assessed student (“you”) through item response choices.

Exhibit 3.17. Sample PISA Collaborative Problem Solving Item

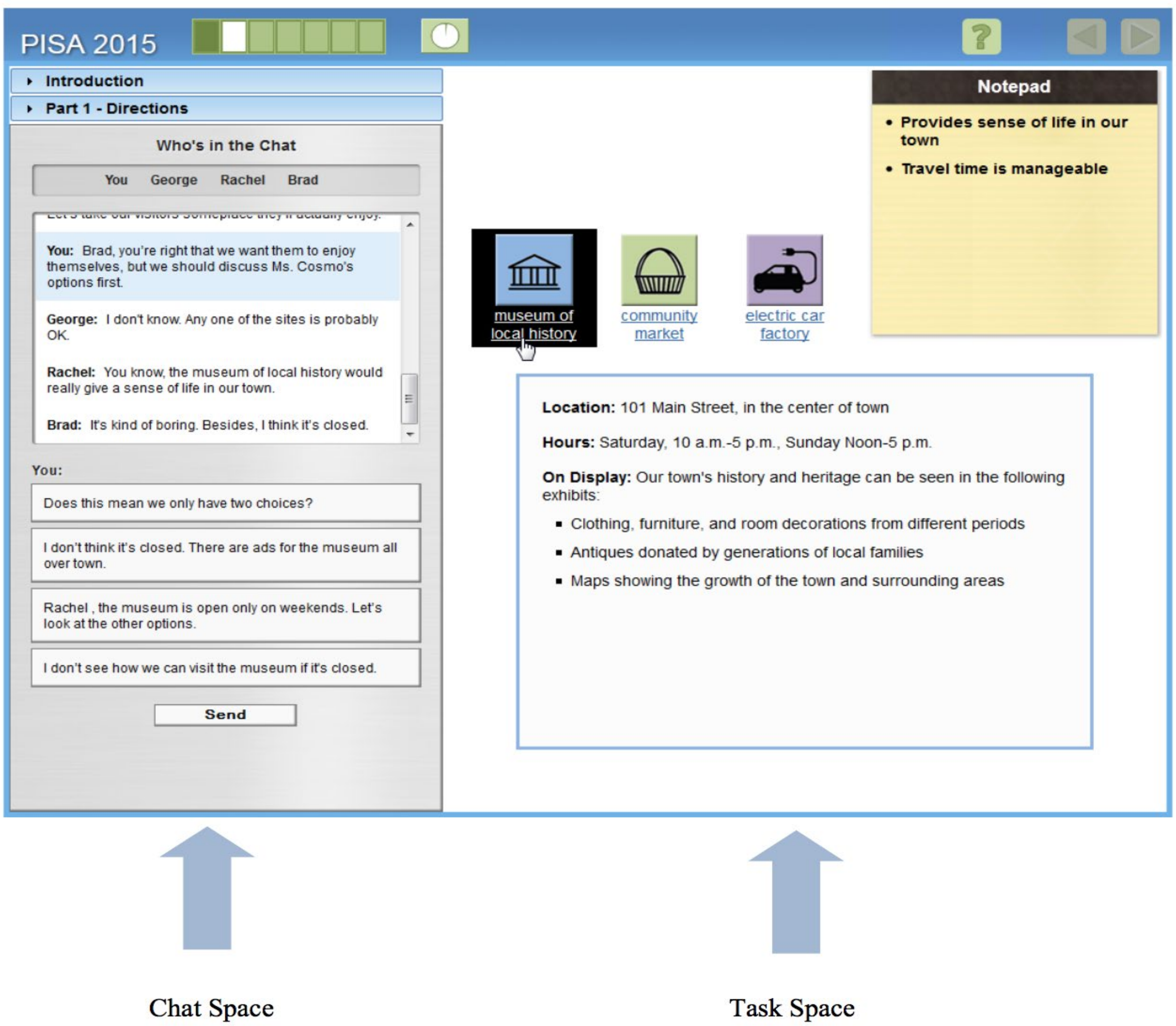
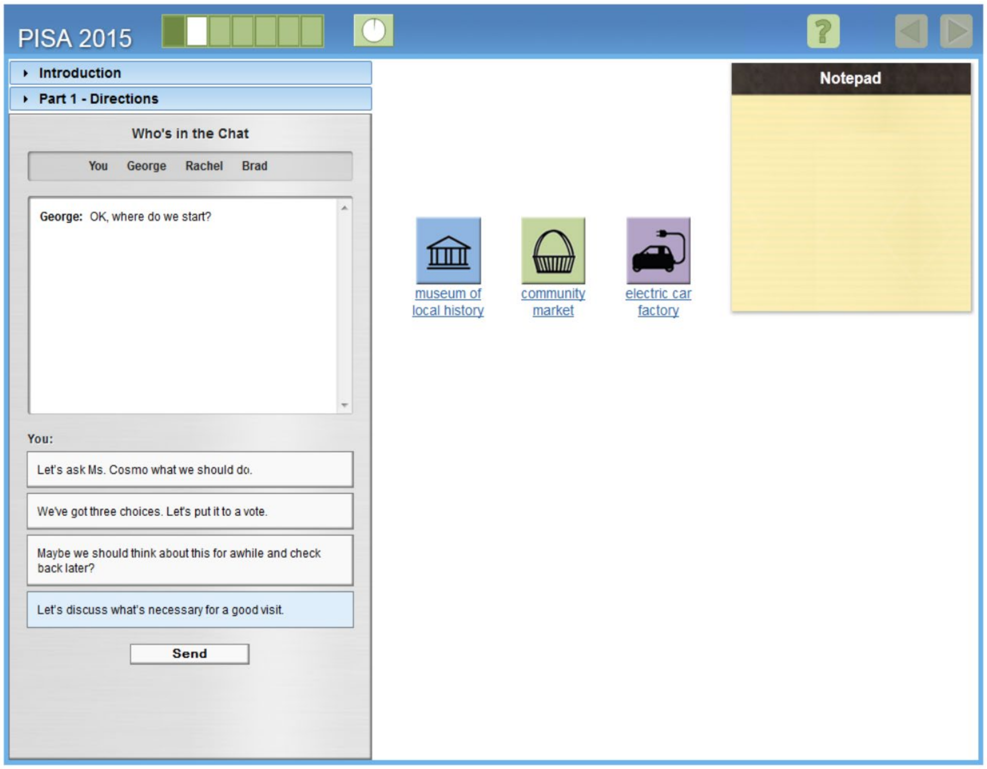
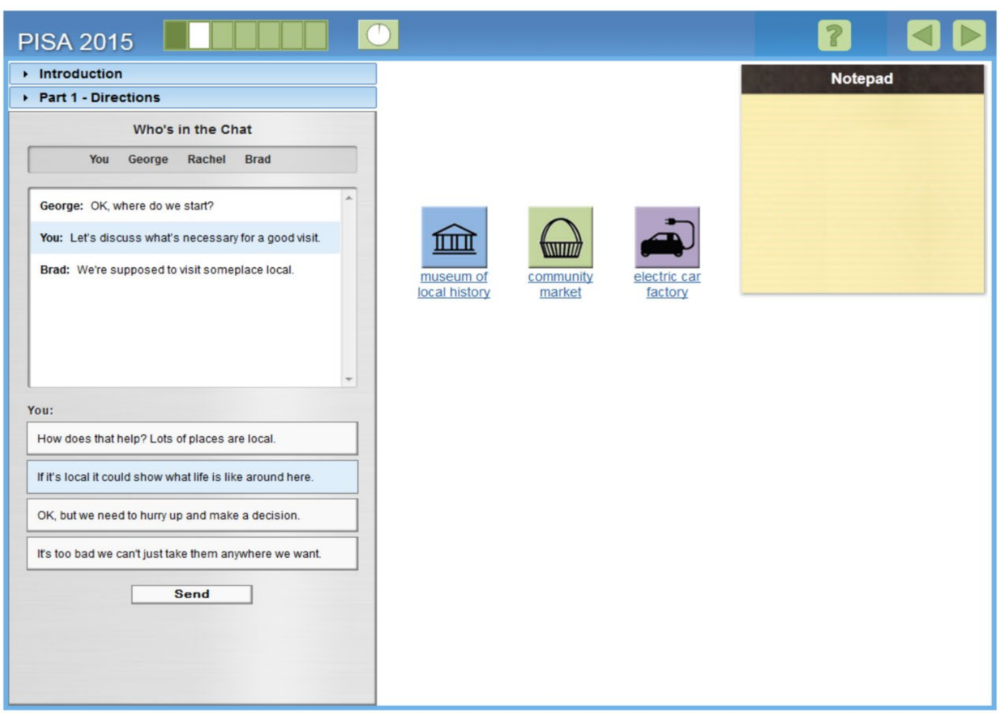


Exhibit 3.18. Example PISA Collaborative Problem-Solving Interaction



Brad mentions that the group is supposed to visit someplace local.



While PISA collaborative problem-solving items are helpful in highlighting discursive assessment, PISA items are not focused on collaborative mathematics in particular. Rather, PISA assesses three collaborative problem-solving competencies: establishing and maintaining a shared understanding; taking appropriate action to solve the problem; and establishing and maintaining team organization. Additionally, PISA's collaborative problem-solving items are intended to assess problem solving competencies such as exploring and understanding; representing and formulating; planning and executing; and monitoring and reflecting.

Some of these competencies apply directly to collaborative mathematics, but the overlap is not complete. The aim for NAEP is to assess the collaborative processes involved in *mathematics* in particular. The following sections describe three measurable skills involved in collaborative mathematics:

- attending to and making sense of the mathematical contributions of others
- evaluating the mathematical merit of the contributions of others
- responding productively to others' mathematical ideas

Collaborative mathematics begins with the sharing of ideas in the form of a conjecture or other contribution that is meant to be communicated to others. A first joint act is made up of both this sharing and how others attend to the conjecture and make sense of it (Forman, Larreamendy-Joerns, Stein, & Brown, 1998). To do so, students must establish a shared understanding about what the problem is and how the problem is being interpreted (Lerman, 1996).

While classroom studies document the importance of making sense of peers' ideas during collaborative mathematics activity, most of the research that focuses on the particular discursive processes involved in making sense of student thinking has looked at teacher talk moves rather than those of students (Chapin, O'Connor, & Anderson, 2009). These moves are nevertheless relevant in framing how students make sense of one another's mathematical thinking. For example, people attend to and make sense of each others' mathematical thinking by *eliciting* and *probing* ideas. Individuals then express and check personal understanding of another's thinking by repeating or *revoicing* the idea (Enyedy, et al., 2008). During a collaborative mathematics assessment task, students can elicit, probe, and revoice peers' ideas as ways to demonstrate and check for understanding.

Revoicing is a particularly powerful discursive opportunity to assess whether a student has understood the mathematical contribution of others. Revoicing is defined as "when one person re-utters another's contribution through the use of repetition, expansion, or rephrasing" (Enyedy, et al, 2008, p. 135). From an assessment perspective, for example, students can be asked to revoice (or put into their own words) the expressed mathematical ideas of another student/avatar, or to justify its mathematical appropriateness.

Once students attend to and make sense of the thinking of others, students must evaluate the mathematical reasonableness of their peer's mathematical contribution. Generally, students express their evaluation of the mathematical reasonableness of an idea through agreement or disagreement, including some explanation or justification for its basis. The expression of agreement or disagreement emerges out of shared understanding (Nathan, Eilam, & Kim, 2007). This skill is critical to the development of productive mathematical argumentation. Experimental and classroom studies have found that students' ideas can be evaluated and become influential due to issues of status or authority rather than mathematics sense-making (Cohen & Lotan, 1997; Engle, Langer-Osuna, & McKinney de Royston, 2014).

Exhibit 3.19 shows a 1992 grade 12 NAEP Mathematics Assessment item suited to assess this particular collaborative skill. In the item, the assessed student is given an exchange by two imagined students, Tracy and Pat. That is, the assessment happens in the context of examining the justifying activity of Pat. Tracy offers a conjecture about which Pat expresses and explains disagreement. The assessed student is asked to evaluate these utterances and decide which is correct and to explain their evaluation. This item is useful because the assessed student has the opportunity to read or hear (through voiceover) Tracy and Pat's own utterances. This conversational format is preferable to items that might offer paraphrased positions that the assessed student is tasked to evaluate.

Exhibit 3.19. Grade 12 Number Properties Collaborative Mathematics Item

Subject: Mathematics, Grade: 12, Year: 1992
Content Classifications: Number properties and operations, Problem solving, Type: SCR, Difficulty Level: Easy

Tracy said, "I can multiply 6 by another number and get an answer that is smaller than 6."

Pat said, "No, you can't. Multiplying 6 by another number always makes the answer 6 or larger."

Who is correct? Give a reason for your answer.

____

____

____

Did you use the calculator on this question?

○ Yes ○ No

Question ID: 1992-12M14 #2 M054801

A third mathematics-specific collective process involves responding productively to others' mathematical ideas. Beyond countering or expressing agreement, students respond to peers' ideas in other mathematically productive ways. In particular, students learn to build on, extend, and connect across mathematical ideas. These discursive acts depend on, and build on, the acts of making sense of and evaluating others' mathematical thinking. Once a shared mathematical idea is understood and taken up, students can further contribute to the mathematical discussion by acting upon those shared ideas.

Connecting across students' mathematical ideas is one such core discursive component of productive collaborative mathematics (Stein, Engle, Smith, & Hughes, 2008). By connecting ideas, students are able to notice and explain how two seemingly different strategies hold the same mathematical ideas (e.g., there is a multiplicative relation between the numerator and denominator of fractions). Students also learn to build on or extend an idea through new examples, next steps, or logical implications.

Challenges

Together, the past several decades of research on mathematics thinking and learning and the consensus judgment of experts in mathematics education call for incorporating mathematical practices into the NAEP Mathematics Assessment. In particular, five mathematical practices are identified: (1) representing; (2) abstracting and generalizing; (3) justifying and proving; (4) mathematical modelling; and (5) collaborative mathematics.

Despite widespread consensus on their importance, there are many challenges to assessing practices. One is the interrelated nature of mathematical practices. Second, while the research literature has focused on mathematics as social activity and on the nature of mathematical practices for several decades, that work draws on a range of theoretical and empirical traditions. There is not consensus on how to define, let alone assess, mathematical practices and this work is still evolving. Finally, given the state of research and item development, it will be challenging to have sufficient numbers of items that assess student achievement with each mathematical practice.

Though these challenges are formidable, they are not insurmountable. They can be addressed as the mathematical practices are incorporated into the 2025 NAEP Mathematics Assessment and refined over successive administrations. It may be that the challenges initially prevent reporting results for each mathematical practice, perhaps only a composite score would be reportable. Nonetheless, NAEP must include explicit attention to mathematical practices and take the lead in designing valid ways to assess the practices and report the results.

Exhibit 3.20. Practices and Content Table
In each cell, practice descriptors are included across the grade levels for a particular content area. The entries in this table are by no means intended to be comprehensive, only exemplary. As is Chapter 2, # denotes an opportunity to assess mathematical literacy. The entries in this table are intended to be illustrative, not comprehensive.

	Number Properties and Operations	Measurement	Geometry	Data Analysis, Statistics, and Probability	Algebra
Representing	**Represent** numbers using visual models (e.g., base 10, number lines, fraction strips). **Use** visual models to compare numbers and as tools to solve problems. **Recognize** and **generate** equivalent expressions for numbers. **Use** array models (e.g., tiles, dots, area) to represent and solve problems. **Create** and **justify** solutions to word problems through numeric representations and operations. # **Recognize**, **apply**, **create,** or **translate** across multiple representations of fractions (e.g., visual models of equivalent fractions) and rational numbers (decimals, fractions, percents). # **Recognize, translate** between, **interpret**, and **compare** written and numerical representations of large numbers. (e.g., thousands, thousandths). **Represent, interpret, or**	# **Select** appropriate units related to representing or measuring an attribute of an object. #**Select** or use appropriate measurement instruments to determine the attributes of an object. # **Convert** between measurement systems (e.g., metric and U.S. customary; currency) # **Convert** between units of measure, (e.g., lengths, time) in the same system.	**Draw** or **sketch** figures from a written description. **Represent** or **describe** figures from different views. **Represent** geometric figures algebraically using coordinates and/or equations. **Visualize** and solve problems using geometry (e.g., using 2-D representations of 3-D objects). **Use** a geometric model of a situation to draw conclusions. # **Represent** problem situations with geometric models to solve mathematical or real-world problems.	For a given set of data, **create** a visual graphical, or tabular representation of the data. # **Compare** and **contrast** different visual and graphical representations of univariate and bivariate data. **Justify** the use of a particular representation of data over another. # **Interpret** visual representations to compare data sets, to draw inferences, or to make conclusions across two or more distinct data sets. **Create** and **use** scatterplots to represent the relationship between two variables and to estimate the strength of the relationship (strong, weak, none).	**Recognize, describe, or extend** numerical and geometric patterns using tables, graphs, words, or symbols. **Use** or **create a** graphical representation of a situation to draw conclusions. **Translate** between different representations of expressions using symbols, graphs, tables, diagrams or written descriptions. **Express** mathematical relationships using equations or inequalities. **Interpret** and **connect** the relationships between symbolic representations of equations and their graphs. **Express** linear and exponential sequences in recursive or explicit forms given a table.

	compare expressions or problem situations involving absolute values.				
Abstracting and Generalizing	**Identify** patterns in numbers, figures, or operations and **generalize** patterns in written, pictorial, or symbolic forms. **Determine** an expression for a recursive pattern **Generalize, describe, or compare** numerical properties and operations across different domains. # **Carry out** operations mentally without performing them. **Extend** a pattern or relationship to a larger set of numbers. **Find** and **generate** structural relationships among sets of numbers.	# **Estimate** or **compare** object size with respect to a given measurement attribute. **Extend** quantified attributes to a larger set. **Select to use** particular measurement characteristics and properties above others. # **Compare** objects with respect to a given attribute, such as length, area, volume, angle measurement, weight, mass, or temperature. **Make connections** between representations of different measurement systems.	**Describe** or **compare** simple properties of, or relationships between, geometric figures. **Reason** with general geometric properties rather than with specific instances. **Identify** common elements across different figures and families of figures (e.g., triangles, quadrilaterals, polygons, polyhedra, etc). **Extend** a geometric relationship from one or more figures to a family of figures.	**Interpret** graphical or tabular representations of data in terms of generalized phenomena (e.g., shape, center, spread, clusters). # **Organize** and display data in order to recognize and make inferences from patterns in the data. **Notice** patterns of outcomes in a probability situation. **Generalize** trends in data to suggest interpretations or infer conclusions.	**Generalize** a pattern appearing in a sequence, table, or graph using words or symbols. **Manipulate** algebraic relationships independent of their contextual references (e.g., solve $F = 9C/5 + 32$ for C). **Use** the structure of an algebraic expression to solve problems. (e.g., solve $(x\text{-}3)^2 = 2$ by taking the square root of each side rather than expanding, simplifying, and solving with the quadratic formula). **Transform** unfamiliar expressions or structures into familiar ones (e.g., move objects into a familiar array). **Identify** commonalities within and across function families. **Develop** general rules for translating functions and graphs **Create** connections across representations. **Determine** rules for functional relationships and **generalize** those rules algebraically.
Justifying and Proving	**Justify** why a numerical relationship or pattern is valid or will always hold.	# **Justify or prove** a claim about physical attributes, comparisons, or measurement properties.	**Justify** relationships of congruence and similarity and apply these relationships using	# **Evaluate** the characteristics of a good survey or of a well-designed experiment and **justify**	**Create, validate** and **justify** conclusions and generalizations about functional relationships.

	Find a counterexample to refute a claim about number properties or operations. * **Prove** numerical relationships through developing deductive arguments, engaging in proof by exhaustion, or employing mathematical induction. **Evaluate** the appropriateness of a provided argument about properties or operations.	# **Explain** why a given attribute can be appropriately measured by the chosen quantity and unit. **Evaluate** the validity of a provided argument making use of measurement. # **Find** a counterexample to disprove a claim about properties such as area, length, or volume.	scaling and proportional reasoning. **Create**, **test** and **validate** geometric conjectures (e.g., distinguish which objects in a collection satisfy a given geometric definition and defend choices). **Analyze** a provided argument about geometric attributes or relationships. **Use** given definitions and theorems to prove geometric conjectures. **Develop** justifications and proofs that rely on a variety of representational modes, including two-column, paragraph, or visual models.	the validity of surveys or experiments. **Justify** or **prove** conjectures about probability and combinatorics. **Create** and **explore** counting arguments in order to develop and justify conjectures.	Use algebraic properties to **develop** a valid mathematical argument. **Verify** a conclusion using algebraic properties. **Prove** algebraic relationships through developing deductive arguments, finding counterexamples, * engaging in proof by exhaustion, and *employing mathematical induction.
Mathematical Modeling	**Use** physical or virtual materials to build a model of a number pattern or to predict or estimate results of a continued pattern. . **Select** and defend an appropriate method of estimation as a model for an estimation problem. **Select** appropriate properties or operations that can be used to build a model of a situation or solve a problem. # **Communicate** and defend a decision about a physical or virtual model involving number and/or operation to an audience for feedback.	#**Identify** the attribute that is appropriate to measure in a given situation. **Select** or use a model unit for an attribute to be measured and defend the use of that unit. **Mathematize** a contextual measurement situation to lead to a solution. # **Determine** and defend the appropriate accuracy of measurement for an object in a problem situation and measure the object to that degree of accuracy.	**Use** existing geometric models to solve mathematical or real-world problems. **Visually Model** the effects of successive (or composite) transformations of figures in the plane **Construct** geometric models using physical or virtual materials to solve mathematical or real-world problems. **Predict** the results of combining, subdividing, and transforming geometric figures.	**Identify** a statistical question to investigate in a given, open-ended or data-rich situation. **Create** or use a statistical model to answer a statistical question or make a prediction about a data set. # **Create** or use a statistical model to assess the validity of a statistical claim. **Create** or use a probability model to calculate or estimate the probability of an event. **Compare** and **contrast** theoretical probabilities (based on sample spaces) with results	**Identify** a mathematical problem from a given situation that could be modeled algebraically. **Identify** the variables needed to create an algebraic model of a situation. # **Write** algebraic relationships, expressions, equations or inequalities to model real world situations. **Revise** an existing algebraic model based on introducing new variables or parameters. **Use** function families to model situations or to solve problems.

		# **Assess** the validity and accuracy of a tool being used to in a measurement task. **Create** a model to convert between two measurement systems. **Construct** scale drawings to be used as measurement models of objects in problem situations		from experimental probabilities (relative frequencies) in a simulation.	#-**Build or apply** a mathematical model of a financial situation. (e.g., a monthly family budget, or a car loan).
Collaborative Mathematics	# **Work with others to express** and **interpret** numbers from real-life contexts. **Build** on a numerical model provided by others to complete a mathematical task **Work with others to explain** and **justify** extensions of patterns to an audience of peers (e.g., revoice the work of others to clarify conjectures about patterns) # **Analyze** the effect of another's estimation method on the accuracy of results. **Reflect** on the work of others to extend a numerical pattern. **Respond productively** to contributions by others and critiques of own work.	# **Work with others** to identify and use appropriate measurement tools or units to complete a task. **Evaluate** the validity of a measurement claim posed by others. Engage in **joint thinking** to reach consensus about a measurement situation **Analyze** others' solutions and **suggest** a critique of their solutions in a situation involving measurement. **Attend** to and **make sense** of the mathematical contributions of others in a situation involving measurement (e.g., revoice the work of others to clarify meaning of choice of measurement units).	**Express** and **justify** agreement or disagreement with a claim made by others in a geometric problem situation. **Attend** to the contributions of others in collaboratively generating a geometric proof. **Build** on the work of others to geometrically model a situation. **Work with others to communicate** geometric arguments to an audience (e.g., revoice the work of others to clarify geometric meanings). **Explain** and **defend** a geometric claim, model, or proof to others. **Evaluate** the merit of others' geometric ideas and productively respond. **Connect** and/or **generalize** across geometric ideas contributed by others in a problem-solving situation.	**Work with others to pose** worthwhile statistical questions given a problem situation or context involving data. **Collaborate with others** across a collection of data sets to construct hypotheses or conclusions. # **Recognize** and critique misleading arguments from data (e.g., from media or other people). **Revoice** the work of others in addressing a statistical or probabilistic situation. **Analyze** the models constructed by others to **evaluate** a new data set. # **Select and make** responses to others to productively move collaboration forward in situations involving data analysis, statistics, or probability.	**Work with others to construct** and **defend** a valid algebraic argument to an audience (e.g., revoice the work of others to clarify assertions about algebraic structure and processes). **Verify** the conclusions of others using algebraic properties. **Construct** and **explain** an algebraic model to an audience. **Seek** and **use** feedback from others in an algebraic situation. # **Discuss** and help form a consensus on an algebraic model (e.g., major variables in a household annual budget; provide mathematically sound advice in response to a personal finance dilemma).

Chapter 4

Overview of the Assessment Design

This chapter provides an overview of the major components of the mathematics assessment design, beginning with a brief description of the 2025 NAEP Mathematics Assessment. This is followed by a discussion of the types of assessment tasks and items and how they can be used to expand the ways in which students are asked to demonstrate what they know and can do in mathematics. In addition, this chapter describes how the assessment should be balanced across the five mathematics content areas described in Chapter 2 and the five mathematical practices given in Chapter 3. This framework intentionally emphasizes increased access for diverse student groups – including English language learners and students with disabilities – to demonstrate their mathematics understanding. Scholarship has demonstrated that students of various ethnic, racial, economic, and cultural backgrounds have salient differences that matter to the format and design of assessment items (Solano-Flores, 2011). In particular, the NAEP Mathematics Assessment will continue to use concepts of universal design for assessment to increase inclusiveness and assessment validity (Thompson, Johnstone, & Thurlow, 2002).

Previous NAEP Mathematics Assessments included discrete items, consisting of selected response and constructed response items. A subset of these were contextual items (e.g., word problems, or modeling and partial modeling tasks). In order for students to demonstrate what they know and can do with respect to the range of mathematics content knowledge and mathematical practices in this framework, the 2025 NAEP Mathematics Assessment includes a new item type: scenario-based tasks. Scenario-based tasks have both context and extended storylines to provide opportunities to demonstrate facility with mathematical practices.

Two fundamental aims motivate the expansion. There is a need to ground the NAEP assessment in relevant tasks and familiar contexts to provide a better measure of student content knowledge and mathematical practices (Eklöf, 2010). Second, by expanding item types and thoughtfully using technology, the NAEP Mathematics Assessment continues to provide greater access to all students, diversifies the ways in which student achievement can be recognized and measured, and more robustly assesses both what students know and what they can do.

Technology provides opportunities, but with each opportunity come myriad constraints and repercussions that must be considered. For example, introducing a new format for items on the NAEP Mathematics Assessment that is interactive or discussion-based, requires that great care be taken to ensure that the design is accessible to students, that students have ample time to understand the way of engaging with the item, and that students have had opportunities to experience the task type. Given the digital divide, as the NAEP Mathematics Assessment evolves, development work should address known and potential implementation challenges and identify ways to mitigate issues of access in doing the assessment that could occur in under-resourced communities (Warschauer, 2016). The NAEP Mathematics Assessment is not to disadvantage students by virtue of the technology features of the assessment.

Types of Tasks, Items, and Supporting Tools

The 2025 NAEP Mathematics Assessment will include existing and new discrete items as well as scenario-based tasks. The following sections begin with descriptions of scenario-based tasks and discrete items in the 2025 NAEP Mathematics Assessment and speak specifically to the role of technology in enhancing these. Next is a discussion of the different types of data gathered during students' response to items. At the end of this section different types of tools available during the assessment are discussed, as well as accessibility.

Scenario-Based Tasks

The goal of scenario-based tasks is to provide evidence of students' ways of knowing and doing mathematics, both independently and collaboratively. For example, the practice of collaborative mathematics can be measured on the 2025 NAEP Mathematics Assessment through student interactions with avatars or artificial intelligence partners. As technology develops, such NAEP Mathematics Assessment interaction could eventually involve collaborative live student groups.

Current and future NAEP Mathematics Assessments can take advantage of evolving digital technologies to create the next generation of scenario-based tasks. Other NAEP frameworks have set a foundation for scenario-based tasks. For example, the 2015 NAEP Science Framework called for the use of interactive computer tasks, as did the 2014 and 2018 NAEP Technology and Engineering Literacy (TEL) Framework. Examples of scenario based tasks from TEL can be found at www.nationsreportcard.gov/tel_2014/#tasks/overview. These existing NAEP assessments provide the language used below.

Interactive scenario-based tasks can elicit rich data, providing evidence of mathematical practices that are difficult to measure with more conventional items and tasks. For example, measuring collaboration has long been a challenge in assessment. Novel methodological approaches have, however, been suggested that use performance outcomes and process data from scenario- and simulation-based collaborative assessment to explore discipline-specific student collaborative activity (Andrews, et al., 2017). These approaches can be used to better assess collaborative mathematics behavior and examine how skill in collaborative mathematics as a practice relates to achievement.

The defining features of the scenarios for the 2025 NAEP Mathematics Assessment are an authentic (for students) context with a motivating question or goal along with item design that supports exploration. A scenario sets a problem-solving context for mathematics activity that – as much as possible – is situated in undertakings that might be performed in society, academic settings, or everyday life. Such scenarios may be well suited to address aspects of mathematical literacy that are present across the mathematics content and mathematical practices described in Chapters 2 and 3. Scenarios may also be especially well-suited to measuring the highly iterative or interactional nature of the mathematical practices described in Chapter 3.

The motivating goal for a scenario might be to solve a particular problem or to complete a certain mission within the scenario. The goal provides the driving rationale for the tasks that the student will perform. It offers a storyline that helps build needed background, define the task's relevance and coherence, and motivates the student to engage with the scenario.

An advantage of digital delivery of the assessment is that scenarios can use multimedia (e.g., images, video, animation; in addition to future technologies) to present the settings for the assessment tasks. As a result, non-mathematical linguistic demand might be reduced while maintaining mathematical rigor. Multimedia can also better scaffold the background understanding that examinees may need to complete a given item. For example, video segments or animations that a student observes, along with text, numbers, and graphics, can convey information necessary for the task to be accomplished. In developing such tasks, related design decisions must be made to serve a particular purpose; nothing should be extraneous or be presented simply for visual interest. While in many cases relevant multimedia content can have positive impact on student engagement and performance, it is also possible that it may introduce competition of attention between visual and auditory channels (Folk, et al., 2015). Cognitive and validity research on multimedia content needs to be conducted and inform design.

Within a scenario, students are given opportunities to select tools from a toolkit and use them to solve problems. Students might be asked, for example, to select a graphing or spreadsheet tool or to use a simulation. Various digital and physical tools may be made available, depending on the scenario. Word-processing with predictive text options, chat/texting, or presentation tools might be available for communication tasks, for example, if deemed relevant to the mathematical understanding being assessed (e.g., in items that target collaborative mathematics).

When designing tools for a scenario, it is necessary to determine which elements of a tool are needed for the activities in the scenario and which features of the tool will be used by students. It is not necessary to provide or simulate a fully featured version of a tool. For example, only certain functions of a spreadsheet tool might be provided that are directly relevant to working on a given item. It would not be necessary to provide all of the other features of the spreadsheet tool. In fact, it would be distracting to students and produce measurement error.

An important consideration for assessment developers when designing scenario-based tasks is to ask what is gained through the selection of a scenario as assessment context. A robust scenario will allow examinees to interact with components of a task in multiple ways, explore alternative outcomes and explanations, find multiple solution paths, and model their thinking. Students could also evaluate the outcomes of the choices they make and convey their understanding about mathematical concepts in diverse ways. For example, a scenario may engage students in a range of mathematical practices and foreground one content area.

Study and piloting of mathematics-specific scenario-based tasks are needed prior to incorporating these tasks into the operational NAEP Mathematics Assessment. As illustrated in the examples in Chapter 3, validated scenario-based tasks that assess collaborative problem solving already exist. In the PISA example in Chapter 3, the task was structured as a dialogue with a collaborative team, made up of avatars and the assessed student in a way that would be nearly impossible to do using only discrete item sets. Those interactive design features could be combined with the kind of scenario and within-task discrete items shown in Exhibit 4.1 (based on grade 8 Stacking Chairs task from the Silicon Valley Mathematics Initiative, 2016).

Exhibit 4.1 Grade 8 Mathematical Modeling Scenario Example

You, Lee, and Pat are the team organizing the spring concert at your school. The school has a large room with a stage but the team will need to arrange for renting chairs from a local company. The chairs must be put in a storage room before the concert. The chairs can be stacked. The team stacked some chairs and measured the heights of the stacks. Below are the notes the team made.

The height of stacked chairs

5 chairs are 51 inches high

3 chairs are 45 inches high

8 chairs are 60 inches high

1. How tall are two chairs stacked together? ________ inches

Lee suggests the chairs be stacked in groups of 10.

2. How tall is a stack of 10 chairs? ________ inches
   Show how you figured it out.

The team decides that groups of 10 chairs will take up too much floor space. The team wants an equation to know how tall a stack will be if you know the number of chairs.

3. Write an equation to find the height, *y*, if the number of chairs in a stack is *x*.

4. Explain how Pat can use the equation you wrote to determine the height of 28 chairs.

The storage room is 15 feet tall. Three feet of space above the stack of chairs is needed (to take chairs off the stack).

5. How many chairs can be in a stack and still fit in the storage room? ______ chairs
   Show how you figured it out.

6. There will be 200 chairs for the audience. What else would the team need to know in order to determine whether or not all 200 chairs will fit in the storage room? Why is the information needed?

Due to their capability to replicate authentic situations (i.e., that students may encounter in their lives) scenarios have the potential to provide a level of accessibility, and support for student engagement with the assessment, that other types of assessment tasks do not. Additionally, scenario-based tasks open up opportunities to simultaneously assess multiple practices or content areas. At the same time, a block of scenario-based tasks may provide less measurement information than a block of discrete items in the same amount of assessment time; scenario-based tasks typically require longer assessment time to reach optimal reliability (Jodoin, 2003).

Scenario-based tasks will occur in sets of short or medium scenarios. Medium scenarios will take students about 20 minutes to complete and the short scenarios will take students at most 10 minutes. Both types of scenarios have common characteristics, but they differ in complexity and in the number of embedded assessment tasks and items to which a student is asked to respond. Longer scenarios are more complex and contain more items.

Response Data and Process Data

A key challenge is the need to capture enough information about mathematics content and mathematical practices for a reliable and valid assessment. When this happens within the context of scenario-based tasks which require more time for engagement and completion, data may be available from fewer items per student. A requirement for future NAEP Mathematics Assessments is to develop validated measures from process data, which is generated based on student interaction with the tools and systems in the scenario-based tasks t (e.g., clickstream or activity logs). The data are different from what might be generated in a non-digital format, so it is necessary to describe how all of the additional process data might be handled.

Conventional items always involve the student in a direct response, which generates *response data*. For example, after being presented with information in a table, the student is asked a text-based question and given a limited set of choices from which to select an answer. Student direct responses can also be used in scenarios. Direct response data can include selection from a set of choices (e.g., multiple choice, checking all of the boxes that apply, or providing a constructed response). Scoring methods for such response data are well established.

By contrast, *process data* measures interactions that the student engages in and may provide relevant evidence about whether the student possesses a skill that is an assessment target. Thus, process data can be captured, measured, and interpreted to generate a score. Clickstream data, activity logs, text, transcribed voice responses are among the ways to capture the state of student activity as they work through a problem. These types of data hold potential power to measure student interactivity in modeling and collaborative mathematics, as well as levels of any mathematical practice (e.g., capturing frequency, density, and intensity of engagement with a mathematical practice or identifying and comparing novice to expert level of a practice through process data). While this capability is powerful in theory, moving from big data sources to carefully constructed and validated measures is difficult to achieve in practice. A special study in the area of mathematics assessment is needed to explore and fully realize the potential of digital, scenario-based tasks.

Discrete Item Types

Discrete items are stand-alone items. These include existing NAEP selected-response and constructed-response items. Central to the development of the 2025 NAEP Mathematics Assessment is the careful selection of ways that students respond in items. Since 1992, the NAEP Mathematics Assessment has used two formats: multiple choice and constructed response. In 2017 the term multiple-choice was revised to “selected response” to account for the wider range of item formats available (e.g., matching) with digitally based assessments. Selected response items require a student to select one or more response options from a given, limited set of choices. In 2025, the NAEP assessment retains selected and constructed response options in discrete items. However, within scenario-based tasks this clear dichotomy becomes blurred. The sets of items in scenario-based tasks are more integrated, and the future availability of process data must be considered. The evolving capabilities of digital technology and the addition of mathematical practices means this framework includes the expansion of the two response types (selected and constructed) to allow for additional tool-based and discourse/collaboration-based responses within scenario-based tasks. Selected response and constructed response items for use on the NAEP Mathematics Assessment include a variety of formats.

Selected Response

- Single-selection multiple choice – Students respond by selecting a single choice from a set of given choices.
- Multiple-selection multiple choice – Students respond by selecting two or more choices that meet the condition stated in the stem of the item.
- Matching – Students respond by inserting (i.e., dragging and dropping) one or more source elements (e.g., a graphic) into target fields (e.g., a table).
- Zones – Students respond by selecting one or more regions on a graphic stimulus.
- Grid – Students evaluate mathematical statements or expressions with respect to certain properties. The answer is entered by selecting cells in a table in which rows typically correspond to the statements and columns to the properties checked.
- In-line choice – Students respond by selecting one option from one or more drop-down menus that may appear in various sections of an item.

A forward-thinking area is in the use of discourse and collaboration responses. These types of items map most directly to the collaborative mathematics and modeling practices outlined in Chapter 3. What might these look like? Current examples ask a learner to interact via a text-based scenario with other characters and choose (e.g., through multiple-choice, limited option selections) from given conversational responses to move the collaborative problem forward. Such a selected response choice then provides some information about the level of collaborative mathematics the learner exhibits. This leads to an selected response type that expands the selected response types listed above to include:

- Discourse/collaboration limited option responses – Students respond by selecting from two or more choices of conversational responses as part of a discourse-based or collaborative task.

Constructed Response

- Fill-in the blank – Students respond by entering a short text in a response box that consists of a single line.
- Extended text – Students respond by entering an extended text in a response box that consists of multiple lines.
- Tool-based responses – Students respond by manipulating or using a tool.

Some selected response items, such as matching or multiple-selection items, have scoring guides to permit partial credit. Every constructed response item has a scoring guide that defines the criteria used to evaluate students’ responses. Some short constructed response items can be scored according to guides that permit partial credit, while others are scored as either correct or incorrect. All constructed response scoring guides are refined from work with a sample of actual student responses gathered during pilot use of items. Students are provided information on elements required for a complete task in some of the individual discrete item stems and/or in overviews of tasks. This provides all students with greater access to the task as well as defines the parameters for their response, honoring their time and energy as they engage in the work.

Students are capable of creating and using a great variety of representations in doing mathematics. Moving forward, it will be important for NAEP Mathematics Assessment developers to continue to expand the ways in which students can represent with digital tools. Tools can allow for formal mathematics representations and symbols, and also allow learners to create and share their own ways of thinking with their own representations. For example, statistical tools such as Tinker Plots and its clones allow students to construct their own graphical representations of data and create their own probability simulators. However, introducing new digital tools must be carefully considered. Familiarity with digital technology in general, and with specific digital tools in particular, can influence student performance (Dunham & Hennessey, 2008). Another potential threat to assessment validity is the accessibility of tools and the affordances for students with and without certain disabilities.

In addition, there is greater ability to capture how learners use manipulatives, both digital on screen and with "smart" physical objects that can monitor activity and be connected to the digital assessment. Here there are at least two opportunities to be forward-thinking. First, further inquiry is warranted into ways to incorporate physical manipulatives that can collect data mapped to practice constructs. The advances in smart tool technology are particularly suited to directly capture the practices outlined in Chapter 3. Second, further work is needed to align the data collected from tasks to valid measures of a construct. For example, one could imagine students manipulating a digital or physical object, and the solution states that they come up with at different points in time (since that is monitored continuously) could provide strong differentiating information about mathematical modeling. A solution state of the physical orientation of an object would be the answer (versus a discrete selection or clicking a multiple-choice option). These – and other opportunities – will help NAEP progress toward the ultimate goal of using tasks in the assessment in ways that capture the variety of ways students know and do mathematics.

Mathematical conversation and collaboration may be assessed more effectively in open-ended constructed response formats. For example, the assessment might ask for and then automatically code responses where learners are asked to explain their thinking or justify a contribution to collaborative mathematics. Note that this technology is not available at the time of this framework revision, but may be by the time of the 2025 NAEP Mathematics Assessment. The assessment might ask learners to input their thinking or dialogue via voice (with automatic transcription into text for coding and analysis), which would dramatically open up ways for learners to demonstrate what they know and can do. Similarly, pairs of students might be asked to turn on an audio documentation (e.g., a recording device) as they work together on a modeling task. The record of discourse would be part of assessment response, measurable evidence of students creating representations, making conjectures, critiquing and debating, revoicing, or justifying their solutions to one another. Considerable research and development work is needed around the technology for natural language processing and related domains, combined with careful mapping to constructs and measurement needs, to realize the aspirational goal of opening up such ways for students to show what they do mathematically. Also, special attention must be considered for consent and privacy when considering any sort of voice recording.

Additional information about the NAEP Mathematics Assessment can be found at www.nagb.gov, nces.ed.gov/nationsreportcard/nqt, and samples of discrete items described in this chapter can be found at https://nces.ed.gov/nationsreportcard/nqt/Home/LegacyGen0Bookmark?subject=mathematics .

NAEP Mathematics Tools

The above sections provide an overview for thinking through – and developing – diverse ways to show what one knows and can do mathematically. Each response type requires related system tools and at times mathematics tools. The digital-based environment of the 2025 NAEP assessment provides the majority of these mathematics tools digitally. All digital NAEP assessments include system tools, which are always available and common across all NAEP assessments. There are also mathematics tools, which are specific to and only available for certain items on NAEP Mathematics Assessment. The materials and accompanying tasks should be carefully chosen to cause minimal disruption of the administration process, and only be provided when relevant to solving the item. Before the assessment, students complete a brief, interactive tutorial designed to teach them about the relevant mathematics tools they will use during the assessment.

The assessment should provide reasonable mathematics tools where possible in measuring students' ability to represent their understandings and to use tools to solve problems. Note that these mathematics tools are only available when relevant to the item. In a digital based environment, students will require tools to enter mathematical expressions, ability to draw, highlight, and erase on the screen, measure the length of virtual objects, plot points on number lines or in coordinate planes, graph lines and functions, create and modify graphical representations, provide computational tools equivalent to a four-function calculator at grade 4, a scientific calculator at grade 8, and a graphing calculator at grade 12. Continuing the policy established for the 2017 digital administration, students will have access to a calculator emulator in blocks of items designated as "calculator blocks." New in 2025 will be the availability of a graphing emulator for grade 12 since high school students typically use graphing calculators or online emulators and not scientific calculators (Crowe & Ma, 2010).

Examples of future digital mathematics tools for the 2025 NAEP Mathematics Assessment may include number tiles, spreadsheets, symbolic algebra manipulators, graphing tools, simulations, and dynamic geometry software. Continued development of mathematics tools (digital, physical, and other) is needed to achieve the goals of more authentic tasks for students and more diverse ways for students to show their knowledge.

Accessibility

The NAEP Mathematics Assessment is designed to measure the achievement of students across the nation. NAEP incorporates inclusive policies and practices into every aspect of the assessment, including selection of students, participation in the assessment administration, and valid and effective accommodations. Regardless of race/ethnicity, socioeconomic status, disability, status as an English language learner, or any other factors, every student has a random chance of being selected, because NAEP is administered to a sample of students who represent the student population of the nation as a whole, and for state level tests, of each individual state. Therefore, NAEP should allow a student to demonstrate mathematical knowledge and skill for students who have learned mathematics in a variety of ways, following different curricula and using different instructional materials; for students who have mastered mathematical content and practices to varying degrees; for students with a variety of disabilities; and for students who are English language learners. The related design issue is to determine a reasonable way to measure mathematics in the same way for students who come to the assessment with different experiences, strengths, and challenges; who approach mathematics from different perspectives; and who have different ways of displaying their knowledge and skill.

Two methods NAEP uses to design an accessible assessment program are developing the standard assessment so that it is accessible and providing accommodations for students with special needs. The first is addressed by careful item and delivery design. For many students with disabilities and students whose native language is not English, the standard administration of the NAEP assessment will be most appropriate. For other students with disabilities and some English language learners, the NAEP mathematics accommodations policy allows for a variety of accommodations, which can be used alone or in combination. Developing engaging scenario-based tasks has the potential to help to involve all students and provide greater access to the assessment for all learners.

Some accommodations are actually built-in features, called Universal Design Elements, of the NAEP system tools. These are available to all students. Other accommodations, such as additional assessment time, are offered for specific students who are eligible for the accommodation The accommodations available in NAEP can be grouped into four categories:

- Some are regarded as Standard NAEP Practice, available in almost all NAEP assessments for SD and ELL students.
- Other accommodations for SD students require special presentation, such as Braille or sign language.
- Other accommodations for ELL students.
- Some accommodations are built-in features of the computer-based assessments that are available to all students and so are referred to as Universal Design Elements.

For more detailed information about item design and accommodations see *Assessment and Item Specifications for the NAEP Mathematics Assessment.*

Matrix Sampling

The design of NAEP uses matrix sampling to enable a broad and deep assessment of how students know and do mathematics that also minimizes the time burden on schools and students. Matrix sampling is a sampling plan in which different samples of students take different samples of items. This means that there are multiple forms of the assessment. Items are distributed so that students taking part in the assessment do not all receive the same items. Matrix sampling greatly increases the capacity to obtain information across a much broader range of the objectives than would otherwise be possible.

Balance of the Assessment

The goal to create an authentic assessment, one based on the experiences of students that includes scenario-based tasks, that will diversify the way that students can show what they know and can do. This vision for 2025 NAEP requires a significant change from the 2017 NAEP Mathematics Assessment. The change poses psychometric challenges. Specifically, scenario-based tasks require more time than discrete items. Given that students have a limited time frame in which to take the assessment, students may not be presented as many items on the new assessment as on previous NAEP versions. Likewise, the emphasis placed on mathematical practices in this framework increases interdependence since multiple practices may be used in the context of particular content-related problems.

The result of the revisions described in this framework is that the 2025 NAEP Mathematics Assessment must maintain an intricate balance of design demands. In particular, four aspects of the assessment that are considered in determining an overall balance are:

- Balance by Mathematics Content
  - Number Problems & Operations
  - Measurement
  - Geometry
  - Data Analysis, Statistics, & Probability
  - Algebra
- Balance by Mathematical Practice
  - Representing
  - Abstracting and Generalizing
  - Justifying and Proving
  - Mathematical Modeling
  - Collaborative Mathematics
- Balance by Task Type
  - Scenarios
  - Discrete items
- Balance by Response Type
  - Selected response
  - Constructed response (short and extended)

Balance of Mathematics Content

Each NAEP Mathematics Assessment item is developed to measure one of the content objectives, which are organized into five major areas. Exhibit 4.2 has distribution of items by grade and content area. See Chapter 2 for more details.

Exhibit 4.2. Percentage Distribution of Items by Grade and Content Area

Content Area	Grade 4	Grade 8	Grade 12
Number Properties and Operations	40	20	10
Measurement	20	15	30
Geometry	15	20	
Data Analysis, Statistics, and Probability	10	20	25
Algebra	15	25	35

Balance of Mathematical Practices

Most of NAEP Mathematics Assessment items will feature at least one of the five mathematical practices (75 to 80 percent). The range of 75 to 80 percent allows flexibility in assessment and item development across grades 4, 8 and 12 while also ensuring that the majority of the assessment is designed to capture information on student knowledge while engaging in mathematical practices. The balance of items (20 to 25 percent), will assess knowledge of content without calling on a particular mathematical practice (e.g., procedural or computational skill). See Exhibit 4.3 for the distribution of items by mathematical practice. Because of the matrix sampling used on the NAEP Mathematics Assessment, the proportions in Exhibit 4.3 are for the entire pool of items used and do not represent the experience of each student.

Exhibit 4.3. Percentage Distribution of Items by Mathematical Practice

Mathematical Practice Area	Percentage of Items
Representing	15
Abstracting and Generalizing	15
Justifying and Proving	25-30
Mathematical Modeling	10
Collaborative Mathematics	10
No practice assessed	20-25

The proportions in Exhibit 4.4 give the approximate time allocation for scenario-based tasks and discrete items across all experiences (i.e., due to matrix sampling design, individual student experience might involve more or less time on each task type in each practice area). As with the distribution of items, the ranges in time allocation allow flexibility in development across grades 4, 8, and 12. Certain formats are likely to be especially valuable in eliciting particular mathematical practices. As illustrated in Chapter 3, discrete items are useful measures of mathematical practices such as representing, abstracting and generalizing, and justifying and proving. Also, as noted in Chapter 3, mathematical modeling and collaborative mathematics are more appropriately measured by scenario-based tasks, as indicated by the larger proportion of assessment time on scenario-based tasks for these practices.

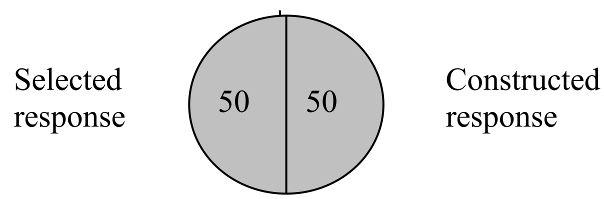
Exhibit 4.4. Percentage of Time Allocation based on Task Type

Mathematical Practice Area	% of Time on Scenario Based Tasks	% of Time on Discrete Items
Representing	30- 40	60-70
Abstracting and Generalizing	20	80
Justifying and Proving	20	80
Mathematical Modeling	75	25
Collaborative Mathematics	100	0
No practice assessed	0	100

Balance by Response Type

As explained in the previous section, discrete items include selected response and constructed response, and these response types may also occur within scenario-based tasks. Selected response includes traditional single-selection multiple choice, as well as other selected-response types such as matching, zones, inline choice, grid, and discourse limited option responses. Constructed response includes short-constructed and extended-constructed response. Types of constructed-response items may include item types such as fill-in-the-blank, extended text, tool-based constructed responses, and discourse and collaboration responses. Testing time on NAEP is divided evenly between selected-response items and constructed-response discrete items as shown below.

Exhibit 4.5. Percent of Testing Time by Response Type



Chapter 5

REPORTING RESULTS OF THE NAEP MATHEMATICS ASSESSMENT

NAEP provides the nation with a snapshot of what U.S. students know and can do in mathematics. Results of the NAEP Mathematics Assessment administrations are reported in terms of average scores for groups of students on the NAEP 0–500 scale and as percentages of students who attain each of the three achievement levels (*NAEP Basic*, *NAEP Proficient*, and *NAEP Advanced*) discussed below. This is an assessment of overall achievement, not a tool for diagnosing the needs of individuals or groups of students. Reported scores are always at the aggregate level; by law, scores are not produced for individual schools or students. Results are reported for the nation as a whole, for regions of the nation, for states, and for large districts that volunteer to participate in the NAEP Trial Urban District Assessment (TUDA). The NAEP results are published in an interactive version online. The online resource provides detailed information on the nature of the assessment, the demographics of the students who participate, and the assessment results.

Legislative Provisions for NAEP Reporting

Under the provisions of the Every Student Succeeds Act (ESSA), states receiving Title I grants must include assurance in their state plans that they will participate in the reading and mathematics state NAEP at grades 4 and 8. Local districts that receive Title I funds must agree to participate in biennial NAEP reading and mathematics administrations at grades 4 and 8 if they are selected to do so. Their results are included in state and national reporting. Participation in NAEP will not substitute for the mandated state-level assessments in reading and mathematics at grades 3 to 8.

In 2002, NAEP initiated TUDA in five large urban school districts that are members of the Council of the Great City Schools (the Atlanta City, City of Chicago, Houston Independent, Los Angeles Unified, and New York City Public Schools districts). In 2003, TUDA began to be administered biennially in odd-numbered years in tandem with state assessments. Sampled students in TUDA districts are assessed in the same subjects and use the same field materials as students selected as part of national main or state samples. TUDA results are reported separately from the state in which the TUDA is located, but results are not reported for individual students or schools. The number of districts participating in TUDA has grown over time to a total of 27 beginning in 2017. With student performance results by district, participating TUDA districts can use results for evaluating their achievement trends and for comparative purposes.

Reporting Scale Scores and Achievement Levels

The NAEP Mathematics Assessment is reported in terms of percentages of students who attain each of the three achievement levels—*NAEP Basic*, *NAEP Proficient*, and *NAEP Advanced* as discussed below. Reported scores are always at the aggregate level. This framework calls for NAEP results to be reported in terms of sub-scores as well, for each content domain and, if feasible, for each mathematical practice. For example, it may be that scores on mathematical practices will be low due to uneven attention to them in mathematics curricula of the past. Sub-scores will allow for a more refined understanding of how students are learning mathematics content and practices that are newly emphasized in recent standards. An overall composite score will also be reported.

Reporting on achievement levels is one way in which NAEP results reach the general public and policymakers. Achievement level results indicate the degree to which student performance meets the standards set for what students should know and be able to do at the *NAEP Basic, NAEP Proficient,* and *NAEP Advanced* levels. Descriptions of achievement levels articulate expectations of performance at each grade level (see Exhibit 5.1). They are reported as percentages of students within each achievement level range, as well as the percentage of students at or above *NAEP Basic* and at or above *NAEP Proficient* ranges. Students performing at or above the *NAEP Proficient* level on NAEP assessments demonstrate solid academic performance and competency over challenging subject matter. It should be noted that the *NAEP Proficient* achievement level does not represent grade level proficiency as determined by other assessment standards (e.g., state or district assessments). Results for students not reaching the *NAEP Basic* achievement level are reported as below *NAEP Basic*. As noted, individual student performance cannot be reported based on NAEP results.

Exhibit 5.1. Generic Achievement Level Policy Definitions for NAEP

Achievement Level	Definition
NAEP Advanced	This level signifies superior performance beyond *NAEP Proficient.*
NAEP Proficient	This level represents solid academic performance for each NAEP assessment. Students reaching this level have demonstrated competency over challenging subject matter, including subject-matter knowledge, application of such knowledge to real-world situations, and analytical skills appropriate to the subject matter.
NAEP Basic	This level denotes partial mastery of prerequisite knowledge and skills that are fundamental for performance at the *NAEP Proficient* level.

Achievement Level Descriptions

Since 1990, the Governing Board has used achievement levels for reporting results on NAEP assessments. The achievement levels represent an informed judgment of “how good is good enough” in the various subjects that are assessed. Generic policy definitions for achievement at the *NAEP Basic, NAEP Proficient,* and *NAEP Advanced* levels describe in very general terms what students at each grade level should know and be able to do on the assessment. Mathematics achievement level descriptions specific to the *2025 NAEP Mathematics Framework* were developed by the Development Panel and can be found in Appendix A1; these will be used to guide item development and initial stages of standard setting for the 2025 NAEP Mathematics Assessment, if it is necessary to conduct a new standard setting.

The content achievement level descriptions may be revised for achievement level setting, if additional information is required. A broadly representative panel of exceptional teachers, educators, and professionals in mathematics will be convened to engage in a standard-setting process to determine the cut scores that correspond to the achievement level descriptions. All achievement level setting activities for NAEP are performed in accordance with current best practices in standard setting and the Governing Board policy on developing student achievement levels for NAEP. The standard setting process does not extend to creating achievement level descriptions for performance that is below the *NAEP Basic* level.

Scoring

Cut scores represent the minimum score required for performance at each NAEP achievement level. Cut scores are reported along with the percentage of students who scored at or above the cut score.

As described in Chapter 4, the design for the 2025 NAEP Mathematics Assessment will include both scenario-based tasks and discrete item types. Items in which there is a single best answer will be scored as correct or incorrect; written responses will be scored using a rubric that evaluates answers according to their match to descriptions in the rubric. Some items will involve process data that documents students' interactions with a task; these interactions can then be assessed for whether they provide relevant evidence about whether the student possesses a skill or understanding. As noted in Chapter 4, clickstream data, activity logs, text, transcribed voice responses, and other data from assessment processes can capture the state of student work as students work through problems. A special study should be conducted to determine how such measures can be productively used in the 2025 NAEP Mathematics Assessment.

Contextual Variables

NAEP law (see section 303(b)(2)(G) of the mandate, https://www.nagb.gov/about-naep/the-naep-law.html) mandates reporting according to various student populations, including:

- Gender,
- Race/ethnicity,
- Eligibility for free/reduced-price lunch,
- Students with disabilities, and
- English language learners.

The National Council on Measurement in Education (NCME) recommends that reports of group differences in assessment performance be accompanied by relevant contextual information, where possible, to enable meaningful interpretation of the differences. That standard reads as follows:

> Reports of group differences in test performance should be accompanied by relevant contextual information, where possible, to enable meaningful interpretation of the differences. If appropriate contextual information is not available, users should be cautioned against misinterpretation. (AERA, 2014, Standard 13.6)

Contextual data about students, teachers, and schools are needed to fulfill the statutory requirement that NAEP include information, whenever feasible, for these groups which promotes meaningful interpretation. Therefore, students, teachers, and school administrators participating in NAEP are asked to respond to questionnaires, which are limited to 15 minutes (computer-based) for students, 20 minutes for teachers, and 30 minutes for each school. Information is also gathered from non-NAEP sources, such as state, district, or school records. For example, the NAEP questionnaires currently include race/ethnicity questions and socio-economic status questions, while gender, age, or other classical "demographic" questions are obtained from school records.

The important components of NAEP reporting are summarized in Exhibit 5.2.

Exhibit 5.2 Components of NAEP Reporting

Component	Key Characteristics
How Information Is Reported	Elements released to the public include: ● Results published mainly online with an interactive report card ● Dedicated website: www.nationsreportcard.gov ● Performance of various subgroups at the national level published in print and online
What Is Reported	NAEP data are reported by: ● Percentage of students attaining achievement levels ● Scale scores ● Sample responses to illustrate achievement level definitions
What Information Is Gathered	Types of background variables distributed to students and schools: ● These are presented in the separate background variables document.

In the past, a range of information has been collected as part of NAEP. In one analysis, Pellegrino et al. (1999) identified five existing categories of indicators: (1) student background characteristics; (2) home and community support for learning; (3) instructional practices and learning resources; (4) teacher education and professional development; and (5) school climate. The categories of information currently collected are:

1. Resources for learning and instruction: people resources, product resources, and time resources
2. Organization of instruction: curriculum content, instructional strategies, use of technology in instruction, and use of formative assessment
3. Teacher preparation: content knowledge and subject-specific training, education and training, professional development, noncognitive teacher factors
4. Student factors: mathematics activities outside of school, self-related beliefs
5. Debrief: student experience with the assessment

Questions do not solicit information about personal topics or information irrelevant to the collection of data on student achievement in mathematics.

The Opportunity Gap

The NAEP Mathematics Framework Development Panel was charged with recommending changes to the subject specific information collected. The Visioning Panel's directive to develop an expansive conception of opportunity to learn (see Exhibit 1.2) is relevant here. Although differences in student achievement have long been referred to as "achievement gaps," scholars have increasingly argued that these differences are likely to also represent gaps in students' opportunities to learn (e.g., Carter & Welner, 2013; Flores, 2007; Martin, 2009; Schmidt et al., 2015). Research has documented the negative effects on achievement of policies and practices that are often found in schools serving the children who live in poverty or have special needs, including an inadequate supply of high quality mathematics teachers with strong knowledge and skills, a tendency to offer few advanced mathematics courses, and a common practice of tracking these students disproportionately into low-demand mathematics courses that restrict their learning opportunities (e.g., Tan & Kastberg, 2017), all of which can be understood as instructional resources that shape what students learn.

Attending to these issues also reflects the sociopolitical turn that has taken place in research on school mathematics, which "highlights mathematics as a dynamic, political, historical, relational, and cultural subject" (TODOS & NCSM, 2016, p.3) in which identity and power both play central roles. This turn has led scholars and educators to explore how school mathematics marginalizes and alienates students who do not see connections to their own lives and experiences. It raises questions about how school mathematics might be reformed to engage all students and their communities. This includes students with disabilities who are often relegated to classrooms where learning disabilities are conceptualized as a deficit rather than a potential strength, and that focus on procedural approaches rather than leveraging students' own particular strategies to successfully engage in mathematical reasoning and sense making (e.g., Lambert et al., 2018). This view, too, is now gaining more traction in research, practice, and policy. These bodies of research informed the Development Panel's conception of opportunities to learn. In particular, when results are interpreted in ways that emphasize achievement gaps without attending to opportunity gaps, differences in subgroups of students can be misinterpreted as differences in student ability, rather than differences due to unequal and inadequate educational opportunities.

Mathematics-Specific Contextual Variables

Contextual variables are selected to be of topical interest, timely, and directly related to academic achievement and current trends and issues in mathematics. As noted in Chapter 1, research has informed an expanded view of the factors that shape opportunities to learn, including time, content, instructional strategies (e.g., how students are grouped for learning; the mathematical tasks they engage in; the opportunities students have to reason, model, and debate ideas), and instructional resources (e.g., the qualifications of their teachers; the material resources available to them; classroom and school policies for the mathematics that students have access to).

Research has demonstrated that what students learn is shaped by the availability of various mathematics programs, curricula, extracurricular activities geared toward mathematics, the percentage of teachers certified in mathematics, teacher years of experience, percentage of mathematics teachers on an emergency license or vacancies/substitute teachers in the school, and number of teachers with mathematics degrees, among other factors. Teachers' and administrators' beliefs about what mathematics is, how one learns mathematics, and who can learn mathematics also affect student learning. What students learn is shaped by their sense of identity and agency. Students who see themselves and who are seen by others as capable mathematical thinkers are more likely to participate in ways that further their learning; students who do not see themselves and are not seen by others as capable mathematical thinkers are likely to be disengaged. Steele, Spencer, and Aronson (2002), for example, found that even passing reminders that a student is a member of one group or another – often in this case a group that is stereotyped as intellectually or academically inferior – can seriously undermine student performance.

Mathematics-specific contextual variables to support reporting for the 2025 NAEP Mathematics Assessment were considered by mapping current items on to the Panel's Opportunity to Learn Framework. These overlap with the OTL strands summarized in Chapter 1, Exhibit 1.2.

Exhibit 5.3. Crosswalk of OTL with Existing Contextual Variables Items

OTL Strand	Resources for learning and instruction	Organization of instruction	Teacher education	Student factors	Debrief
Time (OTL-T)	• time resources				
Content (OTL-C)		• curriculum content			
Instructional Strategies (OTL-IS)		• instructional strategies • use of technology in instruction • use of formative assessment			
Instructional Resources (OTL-IR)	• people resources • product resources	• use of technology in instruction	• content knowledge and subject- specific training • education and training • professional development • noncognitive teacher factors	• mathematics activities outside of school • self-related beliefs • interest and motivation • persistence in mathematics • desire for learning for mathematics	

The Development Panel recommends the following process for revising the mathematics-specific contextual variables. First, existing contextual variable and other information that is collected should be categorized using a frame that includes opportunities to learn (survey items and information collected from other sources). The crosswalk in Exhibit 5.3 suggests that many of the existing items are relevant to this framing. Second, existing survey items should be evaluated in light of changes in technology, policy, and practice. For example, any items about technology use should be edited to reflect current and anticipated changes in the technology used in mathematics classrooms. Items should include reference to both content and practices.

Third, additional items should be developed in the areas of student engagement and identity, views of mathematics teaching and learning, features of classroom instruction, and engagement in mathematics in and out of school. Exhibit 5.4 offers a high-level summary of variable categories. These categories target specific variables within the four opportunity to learn strands identified in Chapter 1 (see Exhibit 1.2).

Exhibit 5.4. Summary Table of Contextual Variable Categories

STUDENT *mathematics-specific sections for Grades 4, 8, and 12*	**TEACHER** *mathematics-specific sections for Grades 4 and 8, and 12*	**ADMINISTRATOR** *mathematics-specific sections for Grades 4, 8, and 12*
Mathematics Content (OTL-C)	Mathematics Content (OTL-C)	School Mathematics Program (OTL-C)
Student Engagement and Identity (OTL-IR)		
Views of Mathematics Teaching and Learning (OTL-IR)	Views of Mathematics Teaching and Learning (OTL-IR)	Views of Mathematics Teaching and Learning (OTL-IR)
Features of Classroom Instruction (OTL-IS)	Features of Classroom Instruction (OTL-IS), including Mathematics Teacher Learning and Support (OTL-IR)	Features of Classroom Instruction (OTL-IS), including Mathematics Teacher Learning and Support (OTL-IR)
Use of Technology (OTL-IS and OTL-IR)	Use of Technology (OTL-IS and OTL-IR)	Use of Technology (OTL-IS and OTL-IR)
Engagement in Mathematics Outside of School (OTL-IR)	Student Engagement in Mathematics Outside of School (OTL-IR)	Student Engagement in Mathematics Outside of School (OTL-IR)
Family Engagement in Mathematics (OTL-IR)	Family Engagement in Mathematics (OTL-IR)	Family Engagement in Mathematics (OTL-IR)

* OTL-C: Content; OTL-IS: Instructional Strategy; OTL-IR: Instructional Resource

Potential changes would include:

1. *Mathematics content*. Because the revised framework conceptualizes mathematics as both content and practices, contextual variables related to mathematics content should be expanded to include reference to mathematical practices as well.
2. *Student engagement and identity*. Questions that prompt student reporting on their perceptions of the importance of mathematics and of their teachers' views of them as mathematics learners should be included. This might include items about whether teachers encourage students to think mathematically, solicit participation in class, or draw on examples of students' cultures and communities to teach mathematics. Ideally, these questions are asked of both students and teachers.
3. *Views of mathematics teaching and learning*. Questions that elicit student, teacher, and administrator views of what it means to learn mathematics, and what it means to teach mathematics, along with questions addressing the development of mathematical identity and agency should be included. These can include items that elicit teachers' views and practices related to the importance of working on challenging problems in learning mathematics or connecting new knowledge to what students already know. Teachers and administrators can be asked about students' funds of knowledge (e.g., in terms of what mathematical strengths students bring with them to the mathematics classroom).
4. *Features of classroom instruction*. Student, teacher, and administrator survey questions about classroom instruction should be revised to include reference to activities associated with the mathematical practices. This would include questions about how often students are asked to explain their thinking, work in pairs or small groups, or use diagrams and mathematical models. Also recommended is an expansive view of instructional materials to include interactive whiteboards, online tools, physical or digital manipulatives, and the like. Questions concerning the supports teachers receive in improving their instruction, including collaborating with peers, working with coaches, and reflecting on student work.
5. *Use of technology.* Existing technology questions should refer more generally to digital devices, while also anticipating questions about the technology that will be present in 2025. In particular, technology questions that specifically address how technology is being used to support mathematics teaching and learning should be considered, including engagement in the mathematical practices.
6. *Engagement in mathematics outside of school*. Student and teacher questions that solicit information about students' engagement with mathematics outside of school should also be added. These might include questions about students' engagement in chess, dominoes, board games, or other kinds of games; experiences working on puzzles such as sudoku or Rubik's cube; use of mathematics in everyday activities like cooking, artwork, playing music, drawing, or taking photographs; their participation in clubs, after school activities, or summer camps for coding or other activities related to computer design and mathematics.
7. *Family engagement in mathematics*. Research has demonstrated the important influence of families on mathematics identity and agency. This includes the beliefs families hold about what it means to do and learn mathematics, who has the capacity to succeed in mathematics, and ways they are able to advocate for and support the mathematics learning of their children—much of which depends on the family's relationship with the school community (Civil, 2007; Civil & Bernier, 2006; Martin, 2006). Questions that

inquire into family views of and engagement in mathematics for students, teachers, and administrators should be considered.

Conclusion

As the Nation's Report Card, NAEP reports on student achievement over time, demonstrating where national progress has (and has not) been made. The NAEP Mathematics Assessment is designed to assess the achievement levels of groups of students through robust and challenging assessments that are well aligned with current understanding of the mathematics content and practices to be learned and that uses technology in ways that maximize both student engagement and accessibility. The results of those assessments are informed by data on contextual variables that illuminate potential differences in opportunities to learn for students.

Based on current research, policy, and practice, the NAEP Mathematics Framework Visioning Panel laid out several major goals: to expand attention to student engagement in reasoning about and doing mathematics, to broaden NAEP's mathematical domains and competencies, to leverage interactive multimedia scenario-based tasks as a way to provide more authentic tasks for students to complete and to increase the assessment's accessibility, and to develop an expansive conception of opportunities to learn that would inform the collection and use of contextual information. Chapters 2 and 3 describe the content and practices of mathematics that students should have access to and demonstrate achievement in. Chapter 4 describes the expansion of the assessment in ways that prudently leverage technology's potential to increase authenticity and accessibility. Chapters 1 and 5 describe an expansive understanding of opportunities to learn, and the role that contextual information plays in meaningful interpretation of the results from future NAEP Mathematics Assessments based on this framework.

The goal of the NAEP Mathematics Assessment is to provide relevant and illuminating data to the nation on what students know and can do in order to improve the chances of all students to achieve their mathematical potential. NAEP scores, illuminated by relevant contextual information, can provide the public, parents, students, and schools useful data on student performance in relation to various achievement levels and demographic subgroups. NAEP does not, however, evaluate results or provide conclusive statements about the level of achievement among the nation's K-12 students. Nor is NAEP designed to inform instruction—to guide how mathematics is taught. It is designed only to measure the performance of a representative sample of U.S. students at the designated grade within the assessment context outlined in this framework.

APPENDIX A1: NAEP MATHEMATICS ACHIEVEMENT LEVELS DESCRIPTIONS

The Achievement Level Descriptions (ALDs) in this appendix provide examples of what students performing at the *NAEP Basic*, *NAEP Proficient*, and *NAEP Advanced* achievement levels should know and be able to do in terms of the mathematics content areas and practices identified in the framework. The intended audiences for these ALDs are the NAEP assessment development contractor and item writers; the ALDs help ensure that a broad range of items is developed at each assessed grade.

Following the ALDs presentation, a set of items for one grade level (grade 8) is included to illustrate the knowledge and skills required at different NAEP achievement levels. The items are not intended to represent the entire set of mathematics content areas or practices, nor do the items imply priority or importance of some content areas or practices above others.

Mathematics Achievement-Levels Descriptions for Grade 4

NAEP Basic	**Fourth-grade students performing at the *NAEP Basic* level should show some evidence of emergent understanding of mathematics concepts and procedures in the five NAEP content areas. Students should show evidence of engagement in the five NAEP practices as detailed below.** Fourth graders performing at the *NAEP Basic* level should be able to estimate and use basic facts to perform simple computations with whole numbers; understand the meaning of fractions and decimals, but not necessarily the relations between fractions and decimals; compare familiar benchmark numbers such as 0, ¼, ½, ⅔, ¾, and 1; name or measure attributes of basic shapes and objects; and solve straightforward problems in all NAEP content areas. Students should be able to represent numbers, shapes, and data using visual models; identify patterns and create visual models; explain or defend their strategy or solution; make mathematical sense of a problem scenario, and select or use visual, physical, or symbolic representations to represent the situation; and share ideas and re-voice the ideas of others.
NAEP Proficient	**Fourth-grade students performing at the *NAEP Proficient* level should be able to recognize when particular concepts, procedures, and strategies are appropriate, and to select, integrate, and apply them to model situations mathematically and solve problems requiring more than the straightforward application of a known procedure or strategy. Students should be able to reason about relationships involving the domains of number, space, or data. Students should show evidence of engagement in the five NAEP practices as detailed below.** Fourth graders performing at the *NAEP Proficient* level should be able to estimate and compute with whole numbers and determine whether results are reasonable.

They should be able to identify, represent, compare, add, and subtract fractions and decimals. Students should be able to identify, describe, and measure basic properties of simple objects and measure or draw angles. Students should be able to represent, read, and interpret a single set of data. Students should be able to recognize, describe, and extend patterns.

Students should be able to create, use, and defend visual models to represent problem situations; abstract or de-contextualize and re-contextualize ideas in routine problems using written and symbolic structures; create arguments, explain why conjectures must be true or demonstrate that they are false, explore with examples or search for counterexamples, understand the role of definitions and counterexamples in mathematical arguments; evaluate arguments; interpret problem situations and choose how to mathematize them (including determining assumptions, posing answerable questions, and determining mathematical representations or symbolizations and tools to use, either created or chosen) and apply the processes to reach a solution; and make sense of and evaluate the mathematical contributions of others through expressing and defending agreement or disagreement.

NAEP Advanced

Fourth-grade students performing at the *NAEP Advanced* level should be able to apply integrated conceptual understanding and procedural knowledge in non-algorithmic ways, such as in complex and nonroutine mathematical or real-world problems in the five NAEP content areas. Students should show evidence of engagement in the five NAEP practices as detailed below.

Fourth graders performing at the *NAEP Advanced* level should be able to solve complex nonroutine real-world problems in all NAEP content areas. These students are expected to draw logical conclusions, justify answers and solution processes by explaining why, as well as how, they were achieved, and generalize patterns.

Students should be able to use, analyze, and justify representations created by others; use structures and patterns to generate a rule and investigate conditions under which the rule applies; use a variety of grade-appropriate proof methods to justify a mathematical statement using valid definitions, statements, theorems, or counterexamples; determine and use a series of processes to mathematize a complex situation and evaluate the results obtained; evaluate the ideas of others and justify their evaluations, as well as extend, connect, or generalize across the ideas of others.

Mathematics Achievement-Levels Descriptions for Grade 8

NAEP Basic

Eighth-grade students performing at the *NAEP Basic* level should show evidence of emergent recognition and application of concepts and procedures to solve problems requiring straightforward application of known procedures or strategies in the five NAEP content areas. Students should show evidence of engagement in the five NAEP practices as detailed below.

Eighth graders performing at the *NAEP Basic* level should be able to solve problems in all NAEP content areas using calculation and strategic reasoning with representations including symbols, words, physical objects, patterns, diagrams, charts, and graphs. This level of performance should signify the capacity to use fundamental concepts in all five domains, to compute with integers and rational numbers, and to handle basic proportional and linear relationships.

Students should be able to represent numbers, shape, and data using visual models; identify patterns and create visual models; explain or defend their strategy or solution; make mathematical sense of a problem scenario and select or use visual, physical or symbolic representations to represent the situation; and share ideas and re-voice the ideas of others.

NAEP Proficient

Eighth-grade students performing at the *NAEP Proficient* level should show evidence of recognizing and applying concepts and procedures to solve problems in the five NAEP content areas requiring more than the straightforward application of a known process or result. They should recognize when particular concepts, procedures, and strategies are appropriate and select, integrate, and apply them to model situations mathematically. Students should be able to reason about relationships involving the domains of number, space, or data. Students should show evidence of engagement in the five NAEP practices as detailed below.

Eighth graders performing at the *NAEP Proficient* level should understand the connections among integers, fractions, percents, and decimals. They should be able to work across these kinds of numbers to examine proportional and linear relationships. They should have a beginning understanding of the representations (language and symbolization) of algebra and linear functions. They should be able to estimate and compare figures or objects with respect to attributes such as length, area, volume, or angle measure. They should be able to identify and justify relationships of congruence, similarity, and symmetry. Students at this level should be able to organize data for analysis and be able to calculate, evaluate, and communicate results within the domain of statistics and probability. They should be able to make appropriate inferences from data and graphs.

Students should be able to create, use, and defend visual models to represent problem situations; abstract or de-contextualize and re-contextualize ideas in routine problems using written and symbolic structures; create arguments, explain why conjectures must be true or demonstrate that they are false, explore with examples or search for counterexamples, understand the role of definitions and counterexamples in mathematical arguments; evaluate arguments; interpret problem situations and choose how to mathematize them (including determining assumptions, posing answerable questions, and determining mathematical representations or symbolizations and tools to use, either created or chosen) and apply the processes to reach a solution; and make sense of and evaluate the mathematical contributions of others through expressing and defending agreement or disagreement.

NAEP Advanced

Eighth-grade students performing at the *NAEP Advanced* level should be able to apply integrated conceptual understanding and procedural knowledge in non-algorithmic ways to complex and non-routine mathematical or real-world problems. They should also be able to justify, generalize, and apply concepts and procedures, and be able to synthesize concepts and processes in the five NAEP content areas. Students should show evidence of engagement in the five NAEP practices as detailed below.

Eighth graders performing at the *NAEP Advanced* level should be able to probe examples and counterexamples in order to shape generalizations from which they can develop models. They should be able to use number sense and geometric awareness to consider the reasonableness of an answer. They should be able to use abstract thinking to create unique problem-solving techniques and explain the reasoning processes underlying their conclusions.

Students should be able to use, analyze, and justify representations created by others; use structures and patterns to generate a rule and investigate conditions under which the rule applies; use a variety of grade-appropriate proof methods to justify a mathematical statement using valid definitions, statements, theorems, or counterexamples; determine and use a series of processes to mathematize a complex situation and evaluate the results obtained; evaluate the ideas of others and justify their evaluations, as well as extend, connect, or generalize across the ideas of others.

Mathematics Achievement-Levels Descriptions for Grade 12

NAEP Basic

Twelfth-grade students performing at the *NAEP Basic* level should exhibit evidence of emergent understanding, recognition, and application of concepts and procedures in the five NAEP content areas. Students should show evidence of engagement in the five NAEP practices as detailed below.

Twelfth graders performing at the *NAEP Basic* level should be able to deal with real numbers in all their forms, common two-and three-dimensional figures, transformations of the plane, coordinate geometry, and basic concepts of probability and statistics. They should also be able to obtain and interpret information about functions presented in various forms, including verbal, graphical, tabular, and symbolic.

Students should be able to represent numbers, shapes, and data using visual models; identify patterns and create visual models; explain or defend their strategy or solution; make mathematical sense of a problem scenario and select or use visual, physical, or symbolic representations to represent the situation; and share ideas and re-voice the ideas of others.

NAEP Proficient

Twelfth-grade students performing at the *NAEP* Proficient level should be able to recognize when particular concepts, procedures, and strategies are appropriate and to select, integrate, and apply them to model situations mathematically to solve problems requiring more than the straightforward application of a known result. Students should be able to reason about relationships involving the domains of number, space, or data. Students should show evidence of engagement in the five NAEP practices as detailed below.

Twelfth-graders performing at the *NAEP Proficient* level students should be able to solve complex non-routine tasks using algebraic and geometric approaches. Students should be able to find, test, and validate geometric and algebraic results and conjectures using a variety of methods. They should be able to design and carry out statistical surveys and experiments and interpret results that are obtained by them or by others. Students should also be able to translate between representations of functions (linear and nonlinear, quadratic and exponential), including verbal, graphical, tabular, and symbolic representations.

Students should be able to create, use, and defend visual models to represent problem situations: abstract or de-contextualize and re-contextualize ideas in routine problems using written and symbolic structures; create arguments, explain why conjectures must be true or demonstrate that they are false, explore with examples or search for counterexamples, understand the role of definitions and counterexamples in mathematical arguments; evaluate arguments; interpret problem situations and choose how to mathematize them (including determining assumptions, posing answerable questions, and determining mathematical representations or symbolizations and tools to use, either created or chosen) and apply the processes to reach a solution; and make sense of and evaluate the mathematical contributions of others through expressing and defending agreement or disagreement.

NAEP Advanced **Twelfth-grade students performing at the *NAEP Advanced* level should demonstrate in-depth knowledge of and be able to reason about mathematical concepts and procedures in the realms of number, algebra, geometry, and statistics. Students should show evidence of engagement in the five NAEP practices as detailed below.**

Twelfth graders performing at the *NAEP Advanced* level should be able to solve complex non-routine tasks using algebraic and geometric approaches and defend their solutions. Students should be able to reason about functions (including transformations) as mathematical objects. They should be able to use properties of functions to analyze relationships and to determine and construct appropriate representations for solving problems. These students should reflect on their reasoning, and they should understand the role of hypotheses, deductive reasoning, and conclusions in geometric proofs and algebraic arguments made by themselves and others. They should be able to design and carry out statistical surveys and experiments using a variety of statistical methods and analyses and interpret results that are obtained by them or by others.

Students should be able to use, analyze, and justify representations created by others; use structures and patterns to generate rules and investigate conditions under which rules apply; use a variety of grade-appropriate proof methods to justify a mathematical statement using valid definitions, statements, theorems, or counterexamples; determine and use a series of processes to mathematize a complex situation and evaluate the results obtained; evaluate the ideas of others and justify their evaluations, as well as extend, connect or generalize across the ideas of others.

Appendix A2: Mathematics Items Illustrating ALDs

NAEP Basic, NAEP Proficient, and NAEP Advanced Achievement Levels for Grade 8

For all the items below, refer to the following Figures 1-3:

Figure 1

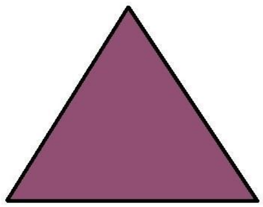


Figure 2

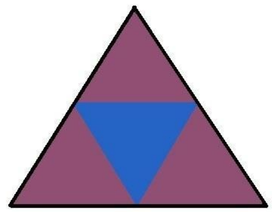
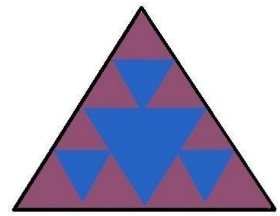


Figure 3



NAEP Basic, Grade 8

Item 1:

Figure 1 is an equilateral triangle, and s is the length of a side of the triangle. If P is the perimeter and A is the area of the triangle in Figure 1, which of the following statements correctly expresses P and A?

a) $P = s$ and $A = \frac{3}{4} s^2$
b) $P = 3s$ and $A = \frac{1}{4} s^2$
c) $P = 3s$ and $A = \sqrt{(\frac{3}{4})}\, s^2$
d) $P = 3s$ and $A = \frac{3}{4}$ s^2
e) $P = s$ and $A = \sqrt{(\frac{3}{4})}\, s^2$

This problem is an indicator of NAEP Basic because students are asked to recognize or apply directly procedures and representations that are expected at grade 8 regarding area and perimeter of triangles.

Item 2:

In Figure 2 the blue triangle has been created by connecting the midpoints of the sides of the original triangle in Figure 1. Indicate if each of the following statements is true or false:

a) The perimeter of the blue triangle is one-fourth the perimeter of the original triangle
b) The perimeter of the blue triangle is one-half the perimeter of the original triangle
c) The area of the blue triangle is one-fourth the area of the original triangle
d) The area of the blue triangle is one-half the area of the original triangle

This problem is an indicator of NAEP Basic because students are asked to recognize/apply simple relationships regarding area and perimeter of triangles.

NAEP PROFICIENT, GRADE 8

Figure 1 is an equilateral triangle, and s is the length of a side of the triangle. In Figure 2 the blue triangle has been created by connecting the midpoints of the sides of the original triangle. In Figure 3 the smaller blue triangles have been created by connecting the midpoints of the sides of each interior triangle in Figure 2.

1) Express the perimeter of the blue triangle in Figure 2 in terms of s.
2) Express the sum of the perimeters of all the blue triangles in Figure 3 in terms of s.

This problem is an indicator of NAEP Proficient because it involves applying a well-known procedure to solve a non-routine problem that should be accessible to grade 8 students and representing the solution using appropriate mathematical representations.

NAEP ADVANCED, GRADE 8

Figure 1 is an equilateral triangle. In Figure 2 the blue triangle has been created by connecting the midpoints of the sides of the original triangle. In Figure 3 the smaller blue triangles have been created by connecting the midpoints of the sides of each interior triangle in Figure 2. Suppose you continue this process of connecting midpoints to obtain subsequent figures (Figure 4, Figure 5, Figure 6, and so on).

1) Express the sum of the perimeters of all the blue triangles in Figure 5 in terms of s.
2) Express the sum of the perimeters of all the blue triangles in Figure 10 in terms of s.

This problem is an indicator of NAEP Advanced because it involves generalizing a pattern and using a well-known procedure in the context of the pattern to solve a non-routine problem, and representing the solution using appropriate mathematical representations.

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The National Assessment Governing Board Charge to the Visioning and Development Panels For the 2025 National Assessment of Educational Progress (NAEP) Mathematics Framework

Whereas, The Nation's Report Card—also known as the National Assessment of Educational Progress (NAEP)—is mandated by Congress to conduct national assessments and report data on student academic achievement and trends in public and private elementary schools and secondary schools, and is prohibited from using any assessment to "evaluate individual students or teachers" or "to establish, require, or influence the standards, assessments, curriculum, … or instructional practices of states or local education agencies" (Public Law 107-279);

Whereas, Congress specifically assigned the National Assessment Governing Board responsibilities to "develop assessment objectives consistent with the requirements of this [law] and test specifications that produce an assessment that is valid and reliable, and are based on relevant widely accepted professional standards";

Whereas, the Governing Board's Strategic Vision adopted in November 2016 established that the Board will, "develop new approaches to update NAEP subject area frameworks to support the Board's responsibility to measure evolving expectations for students, while maintaining rigorous methods that support reporting student achievement trends";

Whereas, the Governing Board established in its Framework Development Policy that the Board shall conduct "a comprehensive, inclusive, and deliberative process" to determine and update the content and format of all NAEP assessments;

Whereas, in accordance with the Governing Board's Framework Development Policy, the Board's Assessment Development Committee conducted a review of the current NAEP Mathematics Framework, which included papers from leading mathematics educators and a comprehensive analysis of current mathematics standards in all 50 states, the District of Columbia, and the Department of Defense Education Activity;

Whereas, based on the review of the NAEP Mathematics Framework conducted by the Assessment Development Committee, the Committee concludes that much of the framework remains relevant, observes that digital platforms and new research encourage innovation in the content and format of future NAEP Mathematics Assessments, and recommends that the Board update the NAEP Mathematics Framework last updated in 2001 "to be informed by a broad, balanced, and inclusive set of factors" balancing "current curricula and instruction, research regarding cognitive development and instruction, and the nation's future needs and desirable levels of achievement, " in accordance with the Framework Development Policy;

Therefore,

- The National Assessment Governing Board staff, with appropriate contractor support and oversight by the Governing Board's Assessment Development Committee, shall conduct a framework update by establishing a Visioning Panel with a subset of members continuing as the Development Panel, in accordance with the Governing Board Framework Development Policy;

- All processes and procedures identified in the Governing Board Framework Development Policy shall be followed;

- The Visioning and Development Panels will recommend to the Board how best to balance necessary changes in the NAEP Mathematics Framework at grades 4, 8, and 12, with the Board's desire for stable reporting of student achievement trends and assessment of a broad range of knowledge and skills, so as to maximize the value of NAEP to the nation; and the Panels are also tasked with considering opportunities to extend the depth of measurement and reporting given the affordances of digital based assessment;

- The update process shall result in three documents: a recommended framework, assessment and item specifications, and recommendations for contextual variables that relate to student achievement in mathematics;

- At the conclusion of the NAEP Mathematics Framework update process, the National Assessment Governing Board shall review recommendations from the Visioning and Development Panels, and take final action on recommended updates to the mathematics framework, assessment specifications, and subject-specific contextual variables; and

- The framework update adopted by the Board will guide development of the 2025 NAEP Mathematics Assessment.